12 CÁLCULO DE PRIMITIVAS

Página 328

Concepto de primitiva

■ **NÚMEROS Y POTENCIAS SENCILLAS**

1. a) $\int 1 = x$ **b)** $\int 2 = 2x$ **c)** $\int \sqrt{2} = \sqrt{2}\,x$

2. a) $\int 2x = x^2$ **b)** $\int x = \frac{x^2}{2}$ **c)** $\int 3x = \frac{3x^2}{2}$

3. a) $\int 7x = \frac{7x^2}{2}$ **b)** $\int \frac{x}{3} = \frac{x^2}{6}$ **c)** $\int \sqrt{2}\,x = \frac{\sqrt{2}\,x^2}{2}$

4. a) $\int 3x^2 = x^3$ **b)** $\int x^2 = \frac{x^3}{3}$ **c)** $\int 2x^2 = \frac{2x^3}{3}$

5. a) $\int 6x^5 = x^6$ **b)** $\int x^5 = \frac{x^6}{6}$ **c)** $\int 3x^5 = \frac{3x^6}{6} = \frac{x^6}{2}$

Página 329

■ **POTENCIAS DE EXPONENTE ENTERO**

6. a) $\int (-1)\,x^{-2} = x^{-1} = \frac{1}{x}$ **b)** $\int x^{-2} = \frac{x^{-1}}{-1} = \frac{-1}{x}$ **c)** $\int \frac{5}{x^2} = \frac{-5}{x}$

7. a) $\int \frac{1}{x^3} = \int x^{-3} = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}$ **b)** $\int \frac{2}{x^3} = 2\int \frac{1}{x^3} = \frac{-2}{2x^2} = \frac{-1}{x^2}$

8. a) $\int \frac{1}{(x-3)^3} = \int (x-3)^{-3} = \frac{(x-3)^{-2}}{-2} = \frac{-1}{2(x-3)^2}$

b) $\int \frac{5}{(x-3)^3} = 5\int \frac{1}{(x-3)^3} = \frac{-5}{2(x-3)^2}$

■ **LAS RAÍCES TAMBIÉN SON POTENCIAS**

9. a) $\int \frac{3}{2}x^{1/2} = x^{3/2} = \sqrt{x^3}$ **b)** $\int \frac{3}{2}\sqrt{x} = \int \frac{3}{2}x^{1/2} = x^{3/2} = \sqrt{x^3}$

10. a) $\int \sqrt{x} = \frac{2}{3}\int \frac{3}{2}x^{1/2} = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3}$ **b)** $\int 7\sqrt{x} = 7\int \sqrt{x} = \frac{14}{3}\sqrt{x^3}$

11. a) $\int \sqrt{3x} = \int \sqrt{3}\sqrt{x} = \sqrt{3}\int \sqrt{x} = \frac{2\sqrt{3}}{3}\sqrt{x^3} = \frac{2\sqrt{3x^3}}{3}$

b) $\int \frac{\sqrt{2x}}{5} = \int \frac{\sqrt{2}}{5}\sqrt{x} = \frac{\sqrt{2}}{5}\int \sqrt{x} = \frac{\sqrt{2}}{5}\cdot\frac{2}{3}\sqrt{x^3} = \frac{2\sqrt{2}}{15}\sqrt{x^3} = \frac{2\sqrt{2x^3}}{15}$

12. a) $\int \frac{1}{2}x^{-1/2} = x^{1/2} = \sqrt{x}$ **b)** $\int \frac{1}{2\sqrt{x}} = \sqrt{x}$

13. a) $\int \frac{3}{2\sqrt{x}} = 3\int \frac{1}{2\sqrt{x}} = 3\sqrt{x}$ **b)** $\int \frac{3}{\sqrt{5x}} = \frac{6}{5}\int \frac{5}{2\sqrt{5x}} = \frac{6}{5}\sqrt{5x}$

14. a) $\int \sqrt{x^3} = \int x^{3/2} = \frac{x^{5/2}}{5/2} = \frac{2}{5}\sqrt{x^5}$ **b)** $\int \sqrt{7x^3} = \sqrt{7}\int \sqrt{x^3} = \frac{2}{5}\sqrt{7x^5}$

¿Recuerdas que D $(ln\ x) = 1/x$?

15. a) $\int \frac{1}{x} = ln\ |x|$ **b)** $\int \frac{1}{5x} = \frac{1}{5}\int \frac{5}{5x} = \frac{1}{5}\ ln\ |5x|$

16. a) $\int \frac{1}{x+5} = ln\ |x+5|$ **b)** $\int \frac{3}{2x+6} = \frac{3}{2}\int \frac{2}{2x+6} = \frac{3}{2}\ ln\ |2x+6|$

Algunas funciones trigonométricas

17. a) $\int cos\ x = sen\ x$ **b)** $\int 2\ cos\ x = 2\ sen\ x$

18. a) $\int cos\left(x + \frac{\pi}{2}\right) = sen\left(x + \frac{\pi}{2}\right)$ **b)** $\int cos\ 2x = \frac{1}{2}\int 2\ cos\ 2x = \frac{1}{2}\ sen\ 2x$

19. a) $\int (-sen\ x) = cos\ x$ **b)** $\int sen\ x = -cos\ x$

20. a) $\int sen(x-\pi) = -cos\ (x-\pi)$ **b)** $\int sen\ 2x = \frac{1}{2}\int 2\ sen\ 2x = \frac{-1}{2}\ cos\ 2x$

21. a) $\int (1 + tg^2\ 2x) = \frac{1}{2}\int 2(1 + tg^2\ 2x) = \frac{1}{2}\ tg\ 2x$

b) $\int tg^2\ 2x = \int (1 + tg^2\ 2x - 1) = \int (1 + tg^2\ 2x) - \int 1 = \frac{1}{2}\ tg\ 2x - x$

■ **ALGUNAS EXPONENCIALES**

22. a) $\int e^x = e^x$ **b)** $\int e^{x+1} = e^{x+1}$

23. a) $\int e^{2x} = \frac{1}{2}\int 2e^{2x} = \frac{1}{2}e^{2x}$ **b)** $\int e^{2x+1} = \frac{1}{2}\int 2e^{2x+1} = \frac{1}{2}e^{2x+1}$

Página 331

1. Calcula las siguientes integrales:

a) $\int 7x^4$ **b)** $\int \frac{1}{x^2}$ **c)** $\int \sqrt{x}$

d) $\int \sqrt[3]{5x^2}$ **e)** $\int \frac{\sqrt[3]{x} + \sqrt{5x^3}}{3x}$ **f)** $\int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}}$

a) $\int 7x^4 = 7\frac{x^5}{5} + k = \frac{7x^5}{5} + k$

b) $\int \frac{1}{x^2} = \int x^{-2} = \frac{x^{-1}}{-1} + k = \frac{-1}{x} + k$

c) $\int \sqrt{x} = \int x^{1/2} = \frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{x^3}}{3} + k$

d) $\int \sqrt[3]{5x^2} = \int \sqrt[3]{5}\, x^{2/3} = \sqrt[3]{5}\,\frac{x^{5/3}}{5/3} + k = \frac{3\sqrt[3]{5x^5}}{5} + k$

e) $\int \frac{\sqrt[3]{x} + \sqrt{5x^3}}{3x} = \int \frac{x^{1/3}}{3x} + \int \frac{\sqrt{5}x^{3/2}}{3x} = \frac{1}{3}\int x^{-2/3} + \frac{\sqrt{5}}{3}\int x^{1/2} =$

$= \frac{1}{3}\,\frac{x^{1/3}}{1/3} + \frac{\sqrt{5}}{3}\,\frac{x^{3/2}}{3/2} + k = \sqrt[3]{x} + \frac{2\sqrt{5x^3}}{9} + k$

f) $\int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}} = \int \frac{\sqrt{5}\cdot x^{3/2}}{\sqrt[3]{3}\cdot x^{1/3}} = \frac{\sqrt{5}}{\sqrt[3]{3}}\int x^{7/6} = \frac{\sqrt{5}}{\sqrt[3]{3}}\,\frac{x^{13/6}}{13/6} + k = \frac{6\sqrt{5}\,\sqrt[6]{x^{13}}}{13\sqrt[3]{3}} + k$

2. Calcula:

a) $\int \frac{x^4 - 5x^2 + 3x - 4}{x}$ **b)** $\int \frac{x^4 - 5x^2 + 3x - 4}{x + 1}$

c) $\int \frac{7x^4 - 5x^2 + 3x - 4}{x^2}$ **d)** $\int \frac{x^3}{x - 2}$

a) $\int \frac{x^4 - 5x^2 + 3x - 4}{x} = \int\left(x^3 - 5x + 3 - \frac{4}{x}\right) = \frac{x^4}{4} - \frac{5x^2}{2} + 3x - 4\ ln\ |x| + k$

b) $\int \frac{x^4 - 5x^2 + 3x - 4}{x + 1} = \int \left(x^3 - x^2 - 4x + 7 - \frac{11}{x + 1} \right) =$

$= \frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 7x - 11\ ln\ |x + 1| + k$

c) $\int \frac{7x^4 - 5x^2 + 3x - 4}{x^2} = \int \left(\frac{7x^4}{x^2} \right) - \int \left(\frac{5x^2}{x^2} \right) + \int \left(\frac{3x}{x^2} \right) - \int \left(\frac{4}{x^2} \right) =$

$= \int 7x^2 - \int 5 + \int \frac{3}{x} - \int \frac{4}{x^2} =$

$= \frac{7x^3}{3} - 5x + 3\ ln\ |x| + \frac{4}{x} + K$

d) $\int \frac{x^3}{x - 2} = \int \left(x^2 + 2x + 4 + \frac{8}{x - 2} \right) = \frac{x^3}{3} + x^2 + 4x + 8\ ln\ |x - 2| + k$

Página 332

3. a) $\int (3x - 5tg\ x) = 3\int x - 5\int tg\ x = \frac{3x^2}{2} - 5\ (-\ ln\ |cos\ x| + k =$

$= \frac{3x^2}{2} + 5\ ln\ |cos\ x| + k$

b) $\int (5\ cos\ x + 3^x) = 5\int cos\ x + \int 3^x = 5\ sen\ x + \frac{3^x}{ln\ 3} + k$

c) $\int (3\ tg\ x - 5\ cos\ x) = 3\int tg\ x - 5\int cos\ x = 3\ (-\ ln|cos\ x|) - 5\ sen\ x + k =$

$= -3\ ln|cosx| - 5\ senx + k$

d) $\int (10^x - 5^x) = \frac{10^x}{ln\ 10} - \frac{5^x}{ln\ 5} + k$

4. a) $\int \frac{3}{x^2 + 1} = 3\ arctg\ x + k$

b) $\int \frac{2x}{x^2 + 1} = ln\ |x^2 + 1| + k$

c) $\int \frac{x^2 - 1}{x^2 + 1} = \int \left(1 + \frac{-2}{x^2 + 1} \right) = x - 2\ arctg\ x + k$

d) $\int \frac{(x + 1)^2}{x^2 + 1} = \int \frac{x^2 + 2x + 1}{x^2 + 1} = \int \left(1 + \frac{2x}{x^2 + 1} \right) = x + ln\ |x^2 + 1| + k$

Página 335

1. Calcula:

a) $\int \cos^4 x \text{ sen } x \, dx$ **b)** $\int 2^{\text{sen } x} \cos x \, dx$

a) $\int \cos^4 x \text{ sen } x \, dx = -\int \cos^4 x (-\text{sen } x) \, dx = -\frac{\cos^5 x}{5} + k$

b) $\int 2^{\text{sen } x} \cos x \, dx = \frac{1}{\ln 2} \int 2^{\text{sen } x} \cos x \cdot \ln 2 \, dx = \frac{2^{\text{sen } x}}{\ln 2} + k$

2. Calcula:

a) $\int \text{cotg } x \, dx$ **b)** $\int \frac{5x}{x^4+1} dx$

a) $\int \text{cotg } x \, dx = \int \frac{\cos x}{\text{sen } x} dx = \ln |\text{sen } x| + k$

b) $\int \frac{5x}{x^4+1} dx = \frac{5}{2} \int \frac{2x}{1+(x^2)^2} dx = \frac{5}{2} \text{arc tg } (x^2) + k$

Página 336

3. Calcula: $\int \frac{1}{\sqrt[3]{x^2} - \sqrt{x}} dx$

Hacemos el cambio $x = t^6$, $dx = 6t^5 \, dt$:

$$\int \frac{1}{\sqrt[3]{x^2} - \sqrt{x}} dx = \int \frac{1}{\sqrt[3]{t^{12}} - \sqrt{t^6}} 6t^5 \, dt = \int \frac{6t^5}{t^4 - t^3} dt = \int \frac{6t^2}{t-1} dt = 6\int \frac{t^2}{t-1} dt =$$

$$= 6\int \left(t + 1 + \frac{1}{t-1}\right) dt = 6\left(\frac{t^2}{2} + t - \ln |t-1|\right) + k =$$

$$= 6\left(\frac{\sqrt[6]{x^2}}{2} + \sqrt[6]{x} - \ln |\sqrt[6]{x} - 1|\right) + k = 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln |\sqrt[6]{x} - 1| + k$$

4. Calcula: $\int \frac{x}{\sqrt{1-x^2}} dx$

Hacemos el cambio $\sqrt{1-x^2} = t \rightarrow 1 - x^2 = t^2 \rightarrow x = \sqrt{1-t^2}$

$$dx = \frac{-t}{\sqrt{1-t^2}} dt$$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{\sqrt{1-t^2}}{t^2} \cdot \frac{-t}{\sqrt{1-t^2}}\,dt = \int -1\,dt = -t + k = -\sqrt{1-x^2} + k$$

Página 336

1. **Calcula:** $\int x \text{ sen } x \, dx$

Llamamos $I = \int x \text{ sen } x \, dx$.

$$\left.\begin{array}{l} u = x, \quad du = dx \\ dv = \text{sen } x \, dx, \quad v = -\cos x \end{array}\right\} \; I = -x \cos x + \int \cos x \, dx = -x \cos x + \text{sen } x + k$$

2. **Calcula:** $\int x \text{ arc tg } x \, dx$

Llamamos $I = \int x \text{ arc tg } x \, dx$.

$$\left.\begin{array}{l} u = \text{arc tg } x, \quad du = \dfrac{1}{1+x^2}\,dx \\ dv = x\,dx, \quad v = \dfrac{x^2}{2} \end{array}\right\}$$

$$I = \frac{x^2}{2} \text{ arc tg } x - \frac{1}{2}\int \left(\frac{x^2}{1+x^2}\right) dx = \frac{x^2}{2} \text{ arc tg } x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx =$$

$$= \frac{x^2}{2} \text{ arc tg } x - \frac{1}{2}[x - \text{arc tg } x] + k = \frac{x^2}{2} \text{ arc tg } x - \frac{1}{2}x + \frac{1}{2} \text{ arc tg } x + k =$$

$$= \frac{x^2+1}{2} \text{ arc tg } x - \frac{1}{2}x + k$$

Página 337

1. **Calcula:** $\int \frac{3x^2 - 5x + 1}{x - 4}\,dx$

$$\int \frac{3x^2 - 5x + 1}{x - 4}\,dx = \int \left(3x + 7 + \frac{29}{x-4}\right) dx = \frac{3x^2}{2} + 7x + 29 \ln |x - 4| + k$$

2. **Calcula:** $\int \frac{3x^2 - 5x + 1}{2x + 1}\,dx$

$$\int \frac{3x^2 - 5x + 1}{2x + 1}\,dx = \int \left(\frac{3}{2}x - \frac{13}{4} + \frac{17/4}{2x+1}\right) dx =$$

$$= \frac{3}{2} \cdot \frac{x^2}{2} - \frac{13}{4}x - \frac{17}{8} \ln |2x + 1| + k = \frac{3x^2}{4} - \frac{13}{4}x - \frac{17}{8} \ln |2x + 1| + k$$

Página 340

3. Calcula:

a) $\displaystyle\int \frac{5x-3}{x^3-x}\,dx$ **b)** $\displaystyle\int \frac{x^2-2x+6}{(x-1)^3}\,dx$

a) Descomponemos la fracción:

$$\frac{5x-3}{x^3-x} = \frac{5x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\frac{5x-3}{x^3-x} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$5x-3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Hallamos A, B y C dando a x los valores 0, 1 y –1:

$$\left.\begin{array}{lllll} x=0 & \Rightarrow & -3=-A & \Rightarrow & A=3 \\ x=1 & \Rightarrow & 2=2B & \Rightarrow & B=1 \\ x=-1 & \Rightarrow & -8=2C & \Rightarrow & C=-4 \end{array}\right\}$$

Así, tenemos que:

$$\int \frac{5x-3}{x^3-x}\,dx = \int\left(\frac{3}{x} + \frac{1}{x-1} - \frac{4}{x+1}\right)dx = 3\,ln|x| + ln|x-1| - 4\,ln|x+1| + k$$

b) Descomponemos la fracción:

$$\frac{x^2-2x+6}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$x^2-2x+6 = A(x-1)^2 + B(x-1) + C$$

Dando a x los valores 1, 0 y 2, queda:

$$\left.\begin{array}{lll} x=1 & \Rightarrow & 5=C \\ x=0 & \Rightarrow & 6=A-B+C \\ x=2 & \Rightarrow & 6=A+B+C \end{array}\right\} \begin{array}{l} A=1 \\ B=0 \\ C=5 \end{array}$$

Por tanto:

$$\int \frac{x^2-2x+6}{(x-1)^3}\,dx = \int\left(\frac{1}{x-1} + \frac{5}{(x-1)^3}\right)dx = ln|x-1| - \frac{5}{2(x-1)^2} + k$$

4. Calcula:

a) $\displaystyle\int \frac{x^3+22x^2-12x+8}{x^4-4x^2}\,dx$ **b)** $\displaystyle\int \frac{x^3-4x^2+4x}{x^4-2x^3-4x^2+8x}\,dx$

a) $x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x-2)(x+2)$

Descomponemos la fracción:

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{x+2}$$

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x-2)(x+2)} =$$

$$= \frac{Ax(x-2)(x+2) + B(x-2)(x+2) + Cx^2(x+2) + Dx^2(x-2)}{x^2(x-2)(x+2)}$$

$$x^3 + 22x^2 - 12x + 8 = Ax(x-2)(x+2) + B(x-2)(x+2) + Cx^2(x+2) + Dx^2(x-2)$$

Hallamos A, B, C y D dando a x los valores 0, 2, –2 y 1:

$$\left.\begin{array}{lllll} x = 0 & \Rightarrow & 8 = -4B & \Rightarrow & B = -2 \\ x = 2 & \Rightarrow & 80 = 16C & \Rightarrow & C = 5 \\ x = -2 & \Rightarrow & 112 = -16D & \Rightarrow & D = -7 \\ x = 1 & \Rightarrow & 19 = -3A - 3B + 3C - D & \Rightarrow & -3A = -9 \Rightarrow A = 3 \end{array}\right\}$$

Por tanto:

$$\int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2}\,dx = \int \left(\frac{3}{x} - \frac{2}{x^2} + \frac{5}{x-2} - \frac{7}{x+2}\right) dx =$$

$$= 3\,ln|x| + \frac{2}{x} + 5\,ln|x-2| - 7\,ln|x+2| + k$$

b) La fracción se puede simplificar:

$$\frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} = \frac{x(x-2)^2}{x(x-2)^2(x+2)} = \frac{1}{x+2}$$

$$\int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x}\,dx = \int \frac{1}{x+2}\,dx = ln|x+2| + k$$

Página 349

EJERCICIOS Y PROBLEMAS PROPUESTOS

PARA PRACTICAR

1 **Calcula las siguientes integrales inmediatas:**

a) $\int (4x^2 - 5x + 7)\,dx$ **b)** $\int \frac{dx}{\sqrt[5]{x}}$ **c)** $\int \frac{1}{2x+7}\,dx$ **d)** $\int (x - sen\,x)\,dx$

a) $\int (4x^2 - 5x + 7)\,dx = \frac{4x^3}{3} - \frac{5x^2}{2} + 7x + k$

b) $\int \frac{dx}{\sqrt[5]{x}} = \int x^{-1/5}\,dx = \frac{x^{4/5}}{4/5} + k = \frac{5\sqrt[5]{x^4}}{4} + k$

c) $\int \frac{1}{2x+7}\,dx = \frac{1}{2}\,ln\,|2x+7| + k$

d) $\int (x - sen\,x)\,dx = \frac{x^2}{2} + cox\,x + k$

2 **Resuelve estas integrales:**

a) $\int (x^2 + 4x)\,(x^2 - 1)\,dx$ **b)** $\int (x-1)^3\,dx$

c) $\int \sqrt{3x}\,dx$ **d)** $\int (sen\,x + e^x)\,dx$

a) $\int (x^2 + 4x)\,(x^2 - 1)\,dx = \int (x^4 + 4x^3 - x^2 - 4x)\,dx = \frac{x^5}{5} + x^4 - \frac{x^3}{3} - 2x^2 + k$

b) $\int (x-1)^3\,dx = \frac{(x-1)^4}{4} + k$

c) $\int \sqrt{3x}\,dx = \int \sqrt{3}\,x^{1/2}\,dx = \sqrt{3}\,\frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{3x^3}}{3} + k$

d) $\int (sen\,x + e^x)\,dx = -cos\,x + e^x + k$

3 **S** **Calcula las integrales siguientes:**

a) $\int {}^{3} \quad dx$ **b)** $\int sen\,(x-4)\,dx$ **c)** $\int \frac{7}{cos^2\,x}\,dx$ **d)** $\int (e^x + 3e^{-x})\,dx$

a) $\int {}^{3} \quad dx = \frac{1}{\sqrt[3]{2}} \int x^{1/3}\,dx = \frac{1}{\sqrt[3]{2}}\,\frac{x^{4/3}}{4/3} + k = \frac{3}{4}\,{}^{3} \quad + k$

b) $\int sen\,(x-4)\,dx = -cos\,(x-4) + k$

c) $\int \frac{7}{cos^2\,x}\,dx = 7\,tg\,x + k$

d) $\int (e^x + 3e^{-x})\,dx = e^x - 3e^{-x} + k$

4 **S** **Halla estas integrales:**

a) $\int \frac{2}{x}\,dx$ **b)** $\int \frac{dx}{x-1}$ **c)** $\int \frac{x + \sqrt{x}}{x^2}\,dx$ **d)** $\int \frac{3}{1+x^2}\,dx$

a) $\int \frac{2}{x}\,dx = 2\,ln\,|x| + k$

b) $\int \frac{dx}{x-1} = ln|x-1| + k$

c) $\int \frac{x+\sqrt{x}}{x^2}\, dx = \int \left(\frac{1}{x} + x^{-3/2}\right) dx = ln|x| - \frac{2}{\sqrt{x}} + k$

d) $\int \frac{3}{1+x^2}\, dx = 3\, arc\, tg\, x + k$

5 **Resuelve las siguientes integrales:**

a) $\int \frac{dx}{x-4}$ **b)** $\int \frac{dx}{(x-4)^2}$ **c)** $\int (x-4)^2 dx$ **d)** $\int \frac{dx}{(x-4)^3}$

a) $\int \frac{dx}{x-4} = ln|x-4| + k$

b) $\int \frac{dx}{(x-4)^2} = \frac{-1}{(x-4)} + k$

c) $\int (x-4)^2 dx = \frac{(x-4)^3}{3} + k$

d) $\int \frac{dx}{(x-4)^3} = \int (x-4)^{-3}\, dx = \frac{(x-4)^{-2}}{-2} + k = \frac{-1}{2(x-4)^2} + k$

6 **Halla las siguientes integrales del tipo exponencial:**

a) $\int e^{x-4}\, dx$ **b)** $\int e^{-2x+9}\, dx$ **c)** $\int e^{5x}\, dx$ **d)** $\int (3^x - x^3)\, dx$

a) $\int e^{x-4}\, dx = e^{x-4} + k$

b) $\int e^{-2x+9}\, dx = \frac{-1}{2}\int -2e^{-2x+9}\, dx = \frac{-1}{2} e^{-2x+9} + k$

c) $\int e^{5x}\, dx = \frac{1}{5}\int 5e^{5x}\, dx = \frac{1}{5} e^{5x} + k$

d) $\int (3^x - x^3)\, dx = \frac{3^x}{ln\, 3} - \frac{x^4}{4} + k$

7 **Resuelve las siguientes integrales del tipo arco tangente:**

a) $\int \frac{dx}{4+x^2}$ **b)** $\int \frac{4\, dx}{3+x^2}$ **c)** $\int \frac{5\, dx}{4x^2+1}$ **d)** $\int \frac{2\, dx}{1+9x^2}$

a) $\int \frac{dx}{4+x^2} = \int \frac{1/4}{1+(x/2)^2}\, dx = \frac{1}{2}\int \frac{1/2}{1+(x/2)^2}\, dx = \frac{1}{2}\, arc\, tg\left(\frac{x}{2}\right) + k$

b) $\int \frac{4\,dx}{3 + x^2} = \int \frac{4/3}{1 + (x/\sqrt{3})^2}\,dx = \frac{4\sqrt{3}}{3} \int \frac{1/\sqrt{3}}{1 + (x/\sqrt{3})^2}\,dx = \frac{4\sqrt{3}}{3}\,arc\ tg\left(\frac{x}{\sqrt{3}}\right) + k$

c) $\int \frac{5\,dx}{4x^2 + 1} = \frac{5}{2} \int \frac{2\,dx}{(2x)^2 + 1} = \frac{5}{2}\,arc\ tg\ (2x) + k$

d) $\int \frac{2\,dx}{1 + 9x^2} = \frac{2}{3} \int \frac{3\,dx}{1 + (3x)^2} = \frac{2}{3}\,arc\ tg\ (3x) + k$

8 **Expresa las siguientes integrales de la forma:**

$$\frac{\textbf{dividendo}}{\textbf{divisor}} = \textbf{cociente} + \frac{\textbf{resto}}{\textbf{divisor}}$$

y resuélvelas:

a) $\int \frac{x^2 - 5x + 4}{x + 1}\,dx$ **b)** $\int \frac{x^2 + 2x + 4}{x + 1}\,dx$ **c)** $\int \frac{x^3 - 3x^2 + x - 1}{x - 2}\,dx$

a) $\int \frac{x^2 - 5x + 4}{x + 1}\,dx = \int \left(x - 6 + \frac{10}{x + 1}\right) dx = \frac{x^2}{2} - 6x + 10\,ln|x + 1| + k$

b) $\int \frac{x^2 + 2x + 4}{x + 1}\,dx = \int \left(x + 1 + \frac{3}{x + 1}\right) dx = \frac{x^2}{2} + x + 3\,ln|x + 1| + k$

c) $\int \frac{x^3 - 3x^2 + x - 1}{x - 2}\,dx = \int \left(x^2 - x - 1 - \frac{3}{x - 2}\right) dx =$

$= \frac{x^3}{3} - \frac{x^2}{2} - x - 3\,ln|x - 2| + k$

9 **Halla estas integrales sabiendo que son del tipo arco seno:**

a) $\int \frac{dx}{\sqrt{1 - 4x^2}}$ **b)** $\int \frac{dx}{\sqrt{4 - x^2}}$ **c)** $\int \frac{e^x}{\sqrt{1 - e^{2x}}}\,dx$ **d)** $\int \frac{dx}{x\sqrt{1 - (ln\ x)^2}}$

a) $\int \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2} \int \frac{2\,dx}{\sqrt{1 - (2x)^2}} = \frac{1}{2}\,arc\ sen\ (2x) + k$

b) $\int \frac{dx}{\sqrt{4 - x^2}} = \int \frac{1/2\,dx}{\sqrt{1 - (x/2)^2}} = arc\ sen\left(\frac{x}{2}\right) + k$

c) $\int \frac{e^x}{\sqrt{1 - e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1 - (e^x)^2}}\,dx = arc\ sen\ (e^x) + k$

d) $\int \frac{dx}{x\sqrt{1 - (ln\ x)^2}} = \int \frac{1/x\,dx}{\sqrt{1 - (ln\ x)^2}} = arc\ sen\ (ln|x|) + k$

10 **Resuelve las integrales siguientes, sabiendo que son de la forma**

$$\int f^n(x) \cdot f'(x):$$

a) $\int \cos x \, sen^3 x \, dx$ **b)** $\int 2x \, e^{x^2} \, dx$ **c)** $\int \frac{x \, dx}{(x^2 + 3)^5}$ **d)** $\int \frac{1}{x} \ln^3 x \, dx$

a) $\int \cos x \, sen^3 x \, dx = \frac{sen^4 x}{4} + k$

b) $\int 2x \, e^{x^2} \, dx = e^{x^2} + k$

c) $\int \frac{x \, dx}{(x^2 + 3)^5} = \frac{1}{2} \int 2x(x^2 + 3)^{-5} \, dx = \frac{1}{2} \frac{(x^2 + 3)^{-4}}{-4} + k = \frac{-1}{8(x^2 + 3)^4} + k$

d) $\int \frac{1}{x} \ln^3 x \, dx = \frac{\ln^4 |x|}{4} + k$

PARA RESOLVER

11 **Resuelve las siguientes integrales:**

a) $\int x^4 \, e^{x^5} \, dx$ **b)** $\int x \, sen \, x^2 \, dx$ **c)** $\int \frac{dx}{\sqrt{9 - x^2}}$ **d)** $\int \frac{x \, dx}{\sqrt{x^2 + 5}}$

a) $\int x^4 \, e^{x^5} \, dx = \frac{1}{5} \int 5x^4 \, e^{x^5} \, dx = \frac{1}{5} e^{x^5} + k$

b) $\int x \, sen \, x^2 \, dx = \frac{1}{2} \int 2x \, sen \, x^2 \, dx = \frac{-1}{2} \cos x^2 + k$

c) $\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{1/3 \, dx}{\sqrt{1 - (x/3)^2}} = arc \, sen \left(\frac{x}{3}\right) + k$

d) $\int \frac{x \, dx}{\sqrt{x^2 + 5}} = \sqrt{x^2 + 5} + k$

Página 350

12 **Resuelve las siguientes integrales:**

a) $\int sen \, x \cos x \, dx$ **b)** $\int \frac{sen \, x \, dx}{\cos^5 x}$ **c)** $\int \sqrt{(x + 3)^5} \, dx$ **d)** $\int \frac{-3x}{2 - 6x^2} \, dx$

a) $\int sen \, x \cos x \, dx = \frac{sen^2 x}{2} + k$

b) $\int \frac{sen\ x\ dx}{cos^5\ x} = -\int (-sen\ x) \cdot cos^{-5}\ x\ dx = \frac{-cos^{-4}\ x}{-4} + k = \frac{1}{4\ cos^4\ x} + k$

c) $\int \sqrt{(x+3)^5}\ dx = \int (x+3)^{5/2}\ dx = \frac{(x+3)^{7/2}}{7/2} + k = \frac{2\sqrt{(x+3)^7}}{7} + k$

d) $\int \frac{-3x}{2-6x^2}\ dx = \frac{1}{4}\int \frac{-12x}{2-6x^2}\ dx = \frac{1}{4}\ ln\,|2-6x^2| + k$

13 **Resuelve las siguientes integrales:**

a) $\int \sqrt{x^2-2x}\,(x-1)\,dx$

b) $\int \frac{arc\ sen\ x}{\sqrt{1-x^2}}\ dx$

c) $\int \frac{(1+ln\ x)^2}{x}\ dx$

d) $\int \sqrt{(1+cos\ x)^3}\ sen\ x\ dx$

a) $\int \sqrt{x^2-2x}\ (x-1)\ dx = \frac{1}{2}\int \sqrt{x^2-2x}\ (2x-2)\ dx = \frac{1}{2}\int (x^2-2x)^{1/2}\ (2x-2)\ dx =$

$$= \frac{1}{2}\ \frac{(x^2-2x)^{3/2}}{3/2} + k = \frac{\sqrt{(x^2-2x)^3}}{3} + k$$

b) $\int \frac{arc\ sen\ x}{\sqrt{1-x^2}}\ dx = \frac{arc\ sen^2\ x}{2} + k$

c) $\int \frac{(1+ln\ x)^2}{x}\ dx = \int (1+ln\ x)^2 \cdot \frac{1}{x}\ dx = \frac{(1+ln\,|x|)^3}{3} + k$

d) $\int \sqrt{(1+cos\ x)^3}\ sen\ x\ dx = -\int (1+cos\ x)^{3/2}\ (-sen\ x)\ dx = -\frac{(1+cos\ x)^{5/2}}{5/2} + k =$

$$= \frac{-2\sqrt{(1+cos\ x)^5}}{5} + k$$

14
S **Aplica la integración por partes para resolver las siguientes integrales:**

a) $\int x\ ln\ x\ dx$

b) $\int e^x cos\ x\ dx$

c) $\int x^2\ sen\ x\ dx$

d) $\int x^2\ e^{2x}\ dx$

e) $\int cos\,(ln\ x)\,dx$

f) $\int x^2\ ln\ x\ dx$

g) $\int arc\ tg\ x\ dx$

h) $\int (x+1)^2\ e^x\ dx$

a) $\int x\ ln\ x\ dx$

$$\begin{cases} u = ln\ x \ \rightarrow\ du = \frac{1}{x}\ dx \\ dv = x\ dx \ \rightarrow\ v = \frac{x^2}{2} \end{cases}$$

$$\int x\ ln\ x\ dx = \frac{x^2}{2}\ ln\ x - \int \frac{x}{2}\ dx = \frac{x^2}{2}\ ln\,|x| - \frac{x^2}{4} + k$$

b) $\int e^x \cos x \, dx$

$$\begin{cases} u = e^x \;\rightarrow\; du = e^x \, dx \\ dv = \cos x \, dx \;\rightarrow\; v = \operatorname{sen} x \end{cases}$$

$$\int e^x \cos x \, dx = e^x \operatorname{sen} x - \underbrace{\int e^x \operatorname{sen} x \, dx}_{I_1}$$

$$\begin{cases} u_1 = e^x \;\rightarrow\; du_1 = e^x \, dx \\ dv_1 = \operatorname{sen} x \, dx \;\rightarrow\; v_1 = -\cos x \end{cases}$$

$$I_1 = -e^x \cos x + \int e^x \cos x \, dx$$

Por tanto:

$$\int e^x \cos x \, dx = e^x \operatorname{sen} x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \operatorname{sen} x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{e^x \operatorname{sen} x + e^x \cos x}{2} + k$$

c) $\int x^2 \operatorname{sen} x \, dx$

$$\begin{cases} u = x^2 \;\rightarrow\; du = 2x \, dx \\ dv = \operatorname{sen} x \, dx \;\rightarrow\; v = -\cos x \end{cases}$$

$$\int x^2 \operatorname{sen} x \, dx = -x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2\underbrace{\int x \cos x \, dx}_{I_1}$$

$$\begin{cases} u_1 = x \;\rightarrow\; du_1 = dx \\ dv_1 = \cos x \, dx \;\rightarrow\; v_1 = \operatorname{sen} x \end{cases}$$

$$I_1 = x \operatorname{sen} x - \int \operatorname{sen} x \, dx = x \operatorname{sen} x + \cos x$$

Por tanto:

$$\int x^2 \operatorname{sen} x \, dx = -x^2 \cos x + 2x \operatorname{sen} x + 2 \cos x + k$$

d) $\int x^2 e^{2x} \, dx$

$$\begin{cases} u = x^2 \;\rightarrow\; du = 2x \, dx \\ dv = e^{2x} \, dx \;\rightarrow\; v = \frac{1}{2} e^{2x} \end{cases}$$

$$\int x^2 e^{2x}\, dx = \frac{x^2}{2} e^{2x} - \underbrace{\int x e^{2x}\, dx}_{I_1}$$

$$\begin{cases} u_1 = x \;\rightarrow\; du_1 = dx \\ dv_1 = e^{2x}\, dx \;\rightarrow\; v_1 = \frac{1}{2} e^{2x} \end{cases}$$

$$I_1 = \frac{x}{2} e^{2x} - \int \frac{1}{2} e^{2x}\, dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x}$$

Por tanto: $\int x^2 e^{2x}\, dx = \frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{4} e^{2x} + k = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) e^{2x} + k$

e) $\int cos\,(ln\, x)\, dx$

$$\begin{cases} u = cos\,(ln\, x) \;\rightarrow\; du = -sen\,(ln\, x) \cdot \frac{1}{x}\, dx \\ dv = dx \;\rightarrow\; v = x \end{cases}$$

$$\int cos\,(ln\, x)\, dx = x\, cos\,(ln\, x) + \underbrace{\int sen\,(ln\, x)\, dx}_{I_1}$$

$$\begin{cases} u_1 = sen\,(ln\, x) \;\rightarrow\; du_1 = cos\,(ln\, x) \cdot \frac{1}{x}\, dx \\ dv_1 = dx \;\rightarrow\; v_1 = x \end{cases}$$

$$I_1 = x\, sen\,(ln\, x) - \int cos\,(ln\, x)\, dx$$

Por tanto:

$$\int cos\,(ln\, x)\, dx = x\, cos\,(ln\, x) + x\, sen\,(ln\, x) - \int cos\,(ln\, x)\, dx$$

$$2\int cos\,(ln\, x)\, dx = x\, cos\,(ln\, x) + x\, sen\,(ln\, x)$$

$$\int cos\,(ln\, x)\, dx = \frac{x\, cos\,(ln\, x) + x\, sen\,(ln\, x)}{2} + k$$

f) $\int x^2\, ln\, x\, dx$

$$\begin{cases} u = ln\, x \;\rightarrow\; du = \frac{1}{x}\, dx \\ dv = x^2\, dx \;\rightarrow\; v = \frac{x^3}{2} \end{cases}$$

$$\int x^2\, ln\, x\, dx = \frac{x^3\, ln\, x}{3} - \int \frac{x^2}{3}\, dx = \frac{x^3\, ln\, x}{3} - \frac{x^3}{9} + k$$

g) $\int arc\ tg\ x\ dx$

$$\begin{cases} u = arc\ tg\ x \ \rightarrow \ du = \dfrac{1}{1+x^2}\,dx \\ dv = dx \ \rightarrow \ v = x \end{cases}$$

$$\int arc\ tg\ x = x\ arc\ tg\ x - \int \frac{1}{1+x^2}\,dx = x\ arc\ tg\ x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx =$$

$$= x\ arc\ tg\ x - \frac{1}{2}\ ln\ (1+x^2) + k$$

h) $\int (x+1)^2\, e^x\, dx$

$$\begin{cases} u = (x+1)^2 \ \rightarrow \ du = 2(x+1)\,dx \\ dv = e^x\,dx \ \rightarrow \ v = e^x \end{cases}$$

$$\int (x+1)^2 e^x\,dx = (x+1)^2 e^x - 2\underbrace{\int (x+1)\,e^x\,dx}_{I_1}$$

$$\begin{cases} u_1 = (x+1) \ \rightarrow \ du_1 = dx \\ dv_1 = e^x\,dx \ \rightarrow \ v_1 = e^x \end{cases}$$

$$I_1 = (x+1)\,e^x - \int e^x\,dx = (x+1)\,e^x - e^x = (x+1-1)\,e^x = x\,e^x$$

Por tanto:

$$\int (x+1)^2 e^x\,dx = (x+1)^2 e^x - 2x\,e^x + k =$$

$$= (x^2 + 2x + 1 - 2x)\,e^x + k = (x^2+1)\,e^x + k$$

15 Calcula $\int cos^4 x\ dx$ utilizando la expresión: $cos^2 x = \dfrac{1}{2} + \dfrac{cos\ 2x}{2}$

$$cos^4 x = \left(\frac{1}{2} + \frac{cos\ 2x}{2}\right)^2 = \frac{1}{4} + \frac{cos^2\ 2x}{4} + \frac{cos\ 2x}{2} =$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{cos\ 4x}{2}\right) + \frac{cos\ 2x}{2} =$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{cos\ 4x}{8} + \frac{cos\ 2x}{2} = \frac{3}{8} + \frac{cos\ 4x}{8} + \frac{cos\ 2x}{2}$$

Por tanto:

$$\int cos^4 x\ dx = \int \left(\frac{3}{8} + \frac{cos\ 4x}{8} + \frac{cos\ 2x}{2}\right) dx = \frac{3}{8}x + \frac{sen\ 4x}{32} + \frac{sen\ 2x}{2} + k$$

16 **Determina el valor de las integrales propuestas en los ejercicios siguientes utilizando la fórmula de integración por partes:**

a) $\int x^2 e^{3x}\, dx$ **b)** $\int \frac{x}{e^x}\, dx$ **c)** $\int 3x \cos x\, dx$ **d)** $\int x^3 \text{ sen } x\, dx$

a) $\int x^2 e^{3x}\, dx$

$$\begin{cases} u = x^2 \rightarrow du = 2x\, dx \\ dv = e^{3x}\, dx \rightarrow v = \frac{1}{3} e^{3x} \end{cases}$$

$$\int x^2 e^{3x}\, dx = \frac{x^2}{3} e^{3x} - \frac{2}{3} \underbrace{\int x\, e^{3x}\, dx}_{I_1}$$

$$\begin{cases} u_1 = x \rightarrow du_1 = dx \\ dv_1 = e^{3x}\, dx \rightarrow v_1 = \frac{1}{3} e^{3x} \end{cases}$$

$$I_1 = \frac{x}{3} e^{3x} - \frac{1}{3} \int e^{3x}\, dx = \frac{x}{3} e^{3x} - \frac{1}{9} e^{3x}$$

Por tanto:

$$\int x^2 e^{3x}\, dx = \frac{x^2}{3} e^{3x} - \frac{2x}{9} e^{3x} + \frac{2}{27} e^{3x} + k = \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} \right) e^{3x} + k$$

b) $\int \frac{x}{e^x}\, dx = \int x\, e^{-x}\, dx$

$$\begin{cases} u = x \rightarrow du = dx \\ dv = e^{-x}\, dx \rightarrow v = -e^{-x} \end{cases}$$

$$\int \frac{x}{e^x}\, dx = -x\, e^{-x} + \int e^{-x}\, dx = -x\, e^{-x} - e^{-x} + k = \frac{-x}{e^x} - \frac{1}{e^x} + k = \frac{-x-1}{e^x} + k$$

c) $\int 3x \cos x\, dx$

$$\begin{cases} u = 3x \rightarrow du = 3\, dx \\ dv = \cos x\, dx \rightarrow v = \text{sen } x \end{cases}$$

$$\int 3x \cos x\, dx = 3x \text{ sen } x - 3 \int \text{sen } x\, dx = 3x \text{ sen } x + 3 \cos x + k$$

d) $\int x^3 \text{ sen } x\, dx$

$$\begin{cases} u = x^3 \rightarrow du = 3x^2\, dx \\ dv = \text{sen } x\, dx \rightarrow v = -\cos x \end{cases}$$

$$\int x^3 \, sen\, x \, dx = -x^3 \cos x + 3 \underbrace{\int x^2 \cos x \, dx}_{I_1}$$

$$\begin{cases} u_1 = x^2 \;\rightarrow\; du_1 = 2x \, dx \\ dv_1 = \cos x \, dx \;\rightarrow\; v_1 = sen\, x \end{cases}$$

$$I_1 = x^2 \, sen\, x - 2 \underbrace{\int x \, sen\, x \, dx}_{I_2}$$

$$\begin{cases} u_2 = x \;\rightarrow\; du_2 = dx \\ dv_2 = sen\, x \, dx \;\rightarrow\; v_2 = -\cos x \end{cases}$$

$$I_2 = -x \cos x + \int \cos x \, dx = -x \cos x + sen\, x$$

Así: $I_1 = x^2 \, sen\, x + 2x \cos x - 2 \, sen\, x$

Por tanto:

$$\int x^3 \, sen\, x \, dx = -x^3 \cos x + 3x^2 \, sen\, x + 6x \cos x - 6 \, sen\, x + k$$

17 **Determina el valor de las integrales que se proponen a continuación:**

a) $\int x \cdot 2^{-x} \, dx$ **b)** $\int arc \cos x \, dx$ **c)** $\int x \cos 3x \, dx$ **d)** $\int x^5 e^{-x^3} \, dx$

a) $\int x \cdot 2^{-x} \, dx$

$$\begin{cases} u = x \;\rightarrow\; du = dx \\ dv = 2^{-x} \, dx \;\rightarrow\; v = \dfrac{-2^{-x}}{ln\, 2} \end{cases}$$

$$\int x \, 2^{-x} \, dx = \frac{-x \cdot 2^{-x}}{ln\, 2} + \int \frac{2^{-x}}{ln\, 2} \, dx = \frac{-x \cdot 2^{-x}}{ln\, 2} + \frac{1}{ln\, 2} \int 2^{-x} \, dx =$$

$$= \frac{-x \cdot 2^{-x}}{ln\, 2} - \frac{2^{-x}}{(ln\, 2)^2} + k$$

b) $\int arc \cos x \, dx$

$$\begin{cases} u = arc \cos x \;\rightarrow\; du = \dfrac{-1}{\sqrt{1 - x^2}} \, dx \\ dv = dx \;\rightarrow\; v = x \end{cases}$$

$$\int arc \cos x \, dx = x \, arc \cos x - \int \frac{-x}{\sqrt{1 - x^2}} \, dx = x \, arc \cos x - \sqrt{1 - x^2} + k$$

c) $\int x \cos 3x \, dx$

$$\begin{cases} u = x \;\rightarrow\; du = dx \\ dv = \cos 3x \, dx \;\rightarrow\; v = \dfrac{1}{3} \operatorname{sen} 3x \end{cases}$$

$$\int x \cos 3x \, dx = \frac{x}{3} \operatorname{sen} 3x - \frac{1}{3} \int \operatorname{sen} 3x \, dx = \frac{x}{3} \operatorname{sen} 3x + \frac{1}{9} \cos 3x + k$$

d) $\int x^5 e^{-x^3} dx = \int \underbrace{x^3}_{u} \cdot \underbrace{x^2 e^{-x^3} dx}_{dv}$

$$\begin{cases} u = x^3 \;\rightarrow\; du = 3x^2 \, dx \\ dv = x^2 e^{-x^3} dx \;\rightarrow\; v = \dfrac{-1}{3} e^{-x^3} \end{cases}$$

$$\int x^5 e^{-x^3} dx = \frac{-x^3}{3} e^{-x^3} + \int x^2 e^{-x^3} dx = \frac{-x^3}{3} e^{-x^3} - \frac{1}{3} e^{-x^3} + k =$$

$$= \frac{(-x^3 - 1)}{3} e^{-x^3} + k$$

18 **En el ejercicio resuelto 7 a), se ha calculado la integral $\int \operatorname{sen}^2 x \, dx$ aplicando la igualdad:**

$$\operatorname{sen}^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$$

Vamos a obtenerla, ahora, mediante la integración por partes, haciendo:

$$\begin{cases} u = \operatorname{sen} x \;\rightarrow\; du = \cos x \, dx \\ dv = \operatorname{sen} x \, dx \;\rightarrow\; v = -\cos x \end{cases}$$

$$\int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + \int \cos^2 x \, dx$$

Si con esta nueva integral procedemos como con la anterior, llegaríamos a una identidad inútil ("se nos va todo"). Compruébalo.

Sin embargo, si hacemos $\cos^2 x = 1 - \operatorname{sen}^2 x$, se resuelve con facilidad. Termina la integral.

- Si aplicáramos el método de integración por partes a la integral $\int \cos^2 x \, dx$, tendríamos que:

$$\begin{cases} u = \cos x \;\rightarrow\; du = -\operatorname{sen} x \, dx \\ dv = \cos x \, dx \;\rightarrow\; v = \operatorname{sen} x \end{cases}$$

Por tanto, quedaría: $\int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + \operatorname{sen} x \cos x + \int \operatorname{sen}^2 x \, dx$

En efecto, es una identidad inútil ("se nos va todo").

• Sin embargo, si hacemos $cos^2 x = 1 - sen^2 x$, tenemos que:

$$\int sen^2 x\, dx = -sen\, x \cos x + \int (1 - sen^2 x)\, dx =$$

$$= -sen\, x \cos x + \int dx - \int sen^2 x\, dx = -sen\, x \cos x + x - \int sen^2 x\, dx$$

Por tanto:

$$2\int sen^2 x\, dx = -sen\, x \cos x + x$$

$$\int sen^2 x\, dx = \frac{-sen\, x \cos x + x}{2} + k = \frac{1}{2}x - \frac{1}{4}\, sen\, 2x + k$$

19 **Determina el valor de las integrales racionales propuestas en los siguientes ejercicios:**

a) $\displaystyle\int \frac{x + 2}{x^2 + 1}\, dx$ **b)** $\displaystyle\int \frac{1}{(x^2 - 1)^2}\, dx$

c) $\displaystyle\int \frac{2x^2 + 7x - 1}{x^3 + x^2 - x - 1}\, dx$ **d)** $\displaystyle\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x}\, dx$

a) $\displaystyle\int \frac{x + 2}{x^2 + 1}\, dx = \frac{1}{2}\int \frac{2x}{x^2 + 1}\, dx + \int \frac{2}{x^2 + 1}\, dx = \frac{1}{2}\, ln\,(x^2 + 1) + 2\, arc\, tg\, x + k$

b) $\displaystyle\int \frac{1}{(x^2 - 1)^2}\, dx = \frac{1}{(x - 1)^2\,(x + 1)^2}\, dx$

Descomponemos en fracciones simples:

$$\frac{1}{(x - 1)^2\,(x + 1)^2} = \frac{A}{(x - 1)} + \frac{B}{(x - 1)^2} + \frac{C}{(x + 1)} + \frac{D}{(x + 1)^2}$$

$$\frac{1}{(x - 1)^2\,(x + 1)^2} = \frac{A(x - 1)(x + 1)^2 + B(x + 1)^2 + C(x + 1)(x - 1)^2 + D(x - 1)^2}{(x - 1)^2\,(x + 1)^2}$$

$$1 = A(x - 1)(x + 1)^2 + B(x + 1)^2 + C(x + 1)(x - 1)^2 + D(x - 1)^2$$

Calculamos A, B, C y D, dando a x los valores 1, –1, 0 y 2:

$$\left.\begin{array}{l} x = 1 \;\to\; 1 = 4B \;\to\; B = 1/4 \\ x = -1 \;\to\; 1 = 4D \;\to\; C = 1/4 \\ x = 0 \;\to\; 1 = -A + B + C + D \;\to\; 1/2 = -A + C \\ x = 2 \;\to\; 1 = 9A + 9B + 3C + D \;\to\; -3/2 = 9A + 3C \;\to\; -1/2 = 3A + C \end{array}\right\} \begin{array}{l} A = -1/4 \\ B = 1/4 \\ C = 1/4 \\ D = 1/4 \end{array}$$

$$\int \frac{1}{(x^2 - 1)^2}\, dx = \int \frac{-1/4}{(x - 1)}\, dx + \int \frac{1/4}{(x - 1)^2}\, dx + \int \frac{1/4}{(x + 1)}\, dx + \int \frac{1/4}{(x + 1)^2}\, dx =$$

$$= \frac{-1}{4}\, ln\,|x - 1| - \frac{1}{4} \cdot \frac{1}{(x + 1)} + \frac{1}{4}\, ln\,|x + 1| - \frac{1}{4} \cdot \frac{1}{(x + 1)} + k =$$

$$= \frac{-1}{4}\left[ln|x-1| + \frac{1}{x-1} - ln|x+1| + \frac{1}{x+1}\right] + k =$$

$$= \frac{-1}{4}\left[ln\left|\frac{x-1}{x+1}\right| + \frac{2x}{x^2-1}\right] + k$$

c) $\displaystyle\int \frac{2x^2+7x-1}{x^3+x^2-x-1}\,dx = \int \frac{2x^2+7x-1}{(x-1)(x+1)^2}\,dx$

Descomponemos en fracciones simples:

$$\frac{2x^2+7x-1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+1^2}$$

$$\frac{2x^2+7x-1}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$2x^2+7x-1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

Hallamos A, B y C:

$$\left.\begin{array}{llll} x=1 & \to\ 8 = 4A & \to & A = 2 \\ x=-1 & \to\ -6 = -2C & \to & C = 3 \\ x=0 & \to\ -1 = A-B-C & \to & B = 0 \end{array}\right\}$$

Por tanto:

$$\int \frac{2x^2+7x-1}{x^3+x^2-x-1}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{3}{(x+1)^2}\,dx = 2\,ln|x-1| - \frac{3}{x+1} + k$$

d) $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx = \int \frac{2x^2+5x-1}{x(x-1)(x+2)}\,dx$

Descomponemos en fracciones simples:

$$\frac{2x^2+5x-1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$\frac{2x^2+5x-1}{x(x-1)(x+2)} = \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)}$$

$$2x^2+5x-1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Hallamos A, B y C:

$$\left.\begin{array}{llll} x=0 & \to\ -1 = -2A & \to & A = 1/2 \\ x=1 & \to\ 6 = 3B & \to & B = 2 \\ x=-2 & \to\ -3 = 6C & \to & C = -1/2 \end{array}\right\}$$

Por tanto:

$$\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x}\, dx = \int \frac{1/2}{x}\, dx + \int \frac{2}{x - 1}\, dx + \int \frac{-1/2}{x + 2}\, dx =$$

$$= \frac{1}{2}\, ln|x| + 2\, ln|x - 1| - \frac{1}{2}\, ln|x + 2| + k = ln\left(\frac{(x - 1)^2\sqrt{x}}{\sqrt{x + 2}}\right) + k$$

20 **Resuelve las siguientes integrales:**

a) $\displaystyle\int \frac{2x - 4}{(x - 1)^2\, (x + 3)}\, dx$ **b)** $\displaystyle\int \frac{2x + 3}{(x - 2)\, (x + 5)}\, dx$

c) $\displaystyle\int \frac{1}{(x - 1)\, (x + 3)^2}\, dx$ **d)** $\displaystyle\int \frac{3x - 2}{x^2 - 4}\, dx$

a) $\displaystyle\int \frac{2x - 4}{(x - 1)^2\, (x + 3)}\, dx$

Descomponemos en fracciones simples:

$$\frac{2x - 4}{(x - 1)^2\, (x + 3)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 3}$$

$$\frac{2x - 4}{(x - 1)^2\, (x + 3)} = \frac{A(x - 1)(x + 3) + B(x + 3) + C(x - 1)^2}{(x - 1)^2\, (x + 3)}$$

$$2x - 4 = A(x - 1)(x + 3) + B(x + 3) + C(x - 1)^2$$

Hallamos A, B y C:

$$\left.\begin{array}{llll} x = 1 & \rightarrow\ -2 = 4B & \rightarrow & B = -1/2 \\ x = -3 & \rightarrow\ -10 = 16C & \rightarrow & C = -5/8 \\ x = 0 & \rightarrow\ -4 = -3A + 3B + C & \rightarrow & A = 5/8 \end{array}\right\}$$

Por tanto:

$$\int \frac{2x - 4}{(x - 1)^2\, (x + 3)}\, dx = \int \frac{5/8}{x - 1}\, dx + \int \frac{-1/2}{(x - 1)^2}\, dx + \int \frac{-5/8}{x + 3}\, dx =$$

$$= \frac{5}{8}\, ln|x - 1| + \frac{1}{2} \cdot \frac{1}{(x - 1)} - \frac{5}{8}\, ln|x + 3| + k = \frac{5}{8}\, ln\left|\frac{x - 1}{x + 3}\right| + \frac{1}{2x - 2} + k$$

b) $\displaystyle\int \frac{2x + 3}{(x - 2)\, (x + 5)}\, dx$

Descomponemos en fracciones simples:

$$\frac{2x + 3}{(x - 2)\, (x + 5)} = \frac{A}{x - 2} + \frac{B}{x + 5} = \frac{A(x + 5) + B(x - 2)}{(x - 2)\, (x + 5)}$$

$$2x + 3 = A(x + 5) + B(x - 2)$$

Hallamos A y B:

$$\left.\begin{array}{llllll} x = 2 & \rightarrow & 7 = 7A & \rightarrow & A = 1 \\ x = -5 & \rightarrow & -7 = -7B & \rightarrow & B = 1 \end{array}\right\}$$

Por tanto:

$$\int \frac{2x + 3}{(x - 2)\,(x + 5)}\,dx = \int \frac{1}{x - 2}\,dx + \int \frac{1}{x + 5}\,dx =$$

$$= ln|x - 2| + ln|x + 5| + k = ln|(x - 2)(x + 5)| + k$$

c) $\displaystyle\int \frac{1}{(x - 1)\,(x + 3)^2}\,dx$

Descomponemos en fracciones simples:

$$\frac{1}{(x - 1)\,(x + 3)^2} = \frac{A}{x - 1} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2}$$

$$\frac{1}{(x - 1)\,(x + 3)^2} = \frac{A(x + 3)^2 + B(x - 1)(x + 3) + C(x - 1)}{(x - 1)\,(x + 3)^2}$$

$$1 = A(x + 3)^2 + B(x - 1)(x + 3) + C(x - 1)$$

Hallamos A, B y C:

$$\left.\begin{array}{lllll} x = 1 & \rightarrow & 1 = 16A & \rightarrow & A = 1/16 \\ x = -3 & \rightarrow & 1 = -4C & \rightarrow & C = -1/4 \\ x = 0 & \rightarrow & 1 = 9A - 3B - C & \rightarrow & B = -1/16 \end{array}\right\}$$

Por tanto:

$$\int \frac{1}{(x - 1)\,(x + 3)^2}\,dx = \int \frac{1/16}{x - 1}\,dx + \int \frac{-1/16}{x + 3}\,dx + \int \frac{-1/4}{(x + 3)^2}\,dx =$$

$$= \frac{1}{16}\,ln|x - 1| - \frac{1}{16}\,ln|x + 3| + \frac{1}{4} \cdot \frac{1}{(x + 3)} + k =$$

$$= \frac{1}{16}\,ln\left|\frac{x - 1}{x + 3}\right| + \frac{1}{4(x + 3)} + k$$

d) $\displaystyle\int \frac{3x - 2}{x^2 - 4}\,dx = \int \frac{3x - 2}{(x - 2)\,(x + 2)}\,dx$

Descomponemos en fracciones simples:

$$\frac{3x - 2}{(x - 2)\,(x + 2)} = \frac{A}{x - 2} + \frac{B}{x + 2} = \frac{A(x + 2) + B(x - 2)}{(x - 2)\,(x + 2)}$$

$$3x - 2 = A(x + 2) + B(x - 2)$$

Hallamos A y B:

$$\left.\begin{array}{llll} x = 2 & \to\ 4 = 4A & \to\ A = 1 \\ x = -2 & \to\ -8 = -4B & \to\ B = 2 \end{array}\right\}$$

Por tanto:

$$\int \frac{3x-2}{x^2-4}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{2}{x+2}\,dx =$$

$$= ln|x-2| + 2\,ln|x+2| + k = ln\left[|x-2|(x+2)^2\right] + k$$

Página 351

21 S **Calcula:**

a) $\displaystyle\int \frac{dx}{x^2-x-2}$

b) $\displaystyle\int \frac{x^4+2x-6}{x^3+x^2-2x}\,dx$

c) $\displaystyle\int \frac{5x^2}{x^3-3x^2+3x-1}\,dx$

d) $\displaystyle\int \frac{2x-3}{x^3-2x^2-9x+18}\,dx$

a) $\displaystyle\int \frac{dx}{x^2-x-2} = \int \frac{dx}{(x+1)(x-2)}$

Descomponemos en fracciones simples:

$$\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

$$1 = A(x-2) + B(x+1)$$

Hallamos A y B:

$$\left.\begin{array}{llll} x = -1 & \to\ 1 = -3A & \to\ A = -1/3 \\ x = 2 & \to\ 1 = 3B & \to\ B = 1/3 \end{array}\right\}$$

Por tanto:

$$\int \frac{dx}{x^2-x-2}\,dx = \int \frac{-1/3}{x+1}\,dx + \int \frac{1/3}{x-2}\,dx =$$

$$= \frac{-1}{3}\,ln|x+1| + \frac{1}{3}\,ln|x-2| + k = \frac{1}{3}\,ln\left|\frac{x-2}{x+1}\right| + k$$

b) $\displaystyle\int \frac{x^4+2x-6}{x^3+x^2-2x}\,dx = \int \left(x - 1 + \frac{3x^2-6}{x(x-1)(x+2)}\right) dx$

Descomponemos en fracciones simples:

$$\frac{3x^2-6}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$\frac{3x^2 - 6}{x(x-1)(x+2)} = \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)}$$

$$3x^2 - 6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

Hallamos A, B y C:

$$\left.\begin{array}{lll} x = 0 & \rightarrow \ -6 = -2A & \rightarrow \ A = 3 \\ x = 1 & \rightarrow \ -3 = 3B & \rightarrow \ B = -1 \\ x = -2 & \rightarrow \ 6 = 6C & \rightarrow \ C = 1 \end{array}\right\}$$

Por tanto:

$$\int \frac{x^4 + 2x - 6}{x^3 + x^2 - 2x}\,dx = \int \left(x - 1 + \frac{3}{x} - \frac{1}{x-1} + \frac{1}{x+2}\right) dx =$$

$$= \frac{x^2}{2} - x + 3\,ln|x| - ln|x-1| + ln|x+2| + k =$$

$$= \frac{x^2}{2} - x + ln\left|\frac{x^3(x+2)}{x-1}\right| + k$$

c) $\displaystyle\int \frac{5x^2}{x^3 - 3x^2 + 3x - 1}\,dx = \int \frac{5x^2}{(x-1)^3}\,dx$

Descomponemos en fracciones simples:

$$\frac{5x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$5x^2 = A(x-1)^2 + B(x-1) + C$$

Hallamos A, B y C:

$$\left.\begin{array}{ll} x = 1 & \rightarrow \ 5 = C \\ x = 2 & \rightarrow \ 20 = A + B + C \\ x = 0 & \rightarrow \ 0 = A - B + C \end{array}\right\} \begin{array}{l} A = 5 \\ B = 10 \\ C = 5 \end{array}$$

Por tanto:

$$\int \frac{5x^2}{x^3 - 3x^2 + 3x - 1}\,dx = \int \left(\frac{5}{x-1} + \frac{10}{(x-1)^2} + \frac{5}{(x-1)^3}\right) dx =$$

$$= 5\,ln|x-1| - \frac{10}{x-1} - \frac{5}{2(x-1)^2} + k$$

d) $\displaystyle\int \frac{2x-3}{x^3 - 2x^2 - 9x + 18}\,dx = \int \frac{2x-3}{(x-2)(x-3)(x+3)}\,dx$

Descomponemos en fracciones simples:

$$\frac{2x-3}{(x-2)(x-3)(x+3)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x+3}$$

$$\frac{2x-3}{(x-2)(x-3)(x+3)} = \frac{A(x-3)(x+3) + B(x-2)(x+3) + C(x-2)(x-3)}{(x-2)(x-3)(x+3)}$$

$$2x - 3 = A(x-3)(x+3) + B(x-2)(x+3) + C(x-2)(x-3)$$

Hallamos A, B y C:

$$\left.\begin{array}{llllll} x = 2 & \rightarrow & 1 = -5A & \rightarrow & A = -1/5 \\ x = 3 & \rightarrow & 3 = 6B & \rightarrow & B = 1/2 \\ x = -3 & \rightarrow & -9 = 30C & \rightarrow & C = -3/10 \end{array}\right\}$$

Por tanto:

$$\int \frac{2x-3}{x^3 - 2x^2 - 9x + 18}\, dx = \int \left(\frac{-1/5}{x-2} + \frac{1/2}{x-3} + \frac{-3/10}{x+3} \right) dx =$$

$$= \frac{-1}{5}\, ln|x-2| + \frac{1}{2}\, ln|x-3| - \frac{3}{10}\, ln|x+3| + k$$

22 **Resuelve las integrales:**

a) $\int \frac{ln\ x}{x}\, dx$ **b)** $\int \frac{1 - sen\ x}{x + cos\ x}\, dx$ **c)** $\int \frac{1}{x\ ln\ x}\, dx$

d) $\int \frac{1 + e^x}{e^x + x}\, dx$ **e)** $\int \frac{sen\ (1/x)}{x^2}\, dx$ **f)** $\int \frac{2x-3}{x+2}\, dx$

g) $\int \frac{arc\ tg\ x}{1 + x^2}\, dx$ **h)** $\int \frac{sen\ x}{cos^4\ x}\, dx$

a) $\int \frac{ln\ x}{x}\, dx = \int \frac{1}{x}\ ln\ x\ dx = \frac{ln^2|x|}{2} + k$

b) $\int \frac{1 - sen\ x}{x + cos\ x}\, dx = ln|x + cos\ x| + k$

c) $\int \frac{1}{x\ ln\ x}\, dx = \int \frac{1/x}{ln\ x}\, dx = ln\left| ln|x| \right| + k$

d) $\int \frac{1 + e^x}{e^x + x}\, dx = ln|e^x + x| + k$

e) $\int \frac{sen\ (1/x)}{x^2}\, dx = -\int \frac{-1}{x^2}\, sen\left(\frac{1}{x}\right) dx = cos\left(\frac{1}{x}\right) + k$

f) $\int \frac{2x-3}{x+2}\, dx = \int \left(2 - \frac{7}{x+2}\right) dx = 2x - 7\ ln|x+2| + k$

g) $\displaystyle\int \frac{arc\ tg\ x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\ arc\ tg\ x\,dx = \frac{arc\ tg^2\ x}{2} + k$

h) $\displaystyle\int \frac{sen\ x}{cos^4\ x}\,dx = -\int (-sen\ x)(cos\ x)^{-4}\,dx = \frac{-(cos\ x)^{-3}}{-3} + k = \frac{1}{3\ cos^3\ x} + k$

23 **Calcula las integrales indefinidas:**

a) $\displaystyle\int \frac{sen\sqrt{x}}{\sqrt{x}}\,dx$ **b)** $\displaystyle\int ln\,(x-3)\,dx$ **c)** $\displaystyle\int \frac{ln\sqrt{x}}{\sqrt{x}}\,dx$

d) $\displaystyle\int ln\,(x^2+1)\,dx$ **e)** $\displaystyle\int (ln\ x)^2\,dx$ **f)** $\displaystyle\int e^x\ cos\ e^x\,dx$

g) $\displaystyle\int \frac{1}{1-x^2}\,dx$ **h)** $\displaystyle\int \frac{(1-x)^2}{1+x}\,dx$

a) $\displaystyle\int \frac{sen\sqrt{x}}{\sqrt{x}}\,dx = -2\int \frac{1}{2\sqrt{x}}\,(-sen\ \sqrt{x})\,dx = -2\ cos\,(\sqrt{x}) + k$

b) $\displaystyle\int ln\,(x-3)\,dx$

$$\begin{cases} u = ln\,(x-3) \ \rightarrow\ du = \dfrac{1}{x-3}\,dx \\ dv = dx \ \rightarrow\ v = x \end{cases}$$

$$\int ln\,(x-3)\,dx = x\ ln|x-3| - \int \frac{x}{x-3}\,dx = x\ ln|x-3| - \int 1 + \frac{3}{x-3}\,dx =$$

$$= x\ ln|x-3| - x - 3\ ln|x-3| + k = (x-3)\ ln|x-3| - x + k$$

c) $\displaystyle\int \frac{ln\sqrt{x}}{\sqrt{x}}\,dx$

$$\begin{cases} u = ln\ \sqrt{x} \ \rightarrow\ du = \dfrac{1}{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2x}\,dx \\ v = \dfrac{1}{\sqrt{x}}\,dx \ \rightarrow\ dv = 2\sqrt{x} \end{cases}$$

$$\int \frac{ln\ \sqrt{x}}{\sqrt{x}}\,dx = 2\sqrt{x}\ ln\ \sqrt{x} - \int \frac{2\sqrt{x}}{2x}\,dx = 2\sqrt{x}\ ln\ \sqrt{x} - \int \frac{1}{\sqrt{x}}\,dx =$$

$$= 2\sqrt{x}\ ln\ \sqrt{x} - 2\sqrt{x} + k = 2\sqrt{x}\,(ln\ \sqrt{x} - 1) + k$$

d) $\int ln\,(x^2+1)\,dx$

$$\begin{cases} u = ln\,(x^2+1) \;\rightarrow\; du = \dfrac{2x}{x^2+1}\,dx \\ dv = dx \;\rightarrow\; v = x \end{cases}$$

$$\int ln\,(x^2+1)\,dx = x\,ln\,(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx =$$

$$= x\,ln\,(x^2+1) - \int \left(2 - \frac{2}{x^2+1}\right) dx = x\,ln\,(x^2+1) - 2x + 2\,arc\,tg\,x + k$$

e) $\int (ln\,x)^2\,dx$

$$\begin{cases} u = (ln\,x)^2 \;\rightarrow\; du = 2\,(ln\,x)\cdot\dfrac{1}{x}\,dx \\ dv = dx \;\rightarrow\; v = x \end{cases}$$

$$\int (ln\,x)^2\,dx = x\,(ln\,x)^2 - 2\int ln\,x\,dx = x\,ln^2|x| - 2\,x\,ln|x| + 2x + k$$

f) $\int e^x\,cos\,e^x\,dx = sen\,e^x + k$

g) $\int \dfrac{1}{1-x^2}\,dx = \int \dfrac{-1}{(x+1)(x-1)}\,dx$

Descomponemos en fracciones simples:

$$\frac{-1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

Hallamos A y B:

$$\left.\begin{array}{l} x = -1 \;\rightarrow\; -1 = -2A \;\rightarrow\; A = 1/2 \\ x = 1 \;\rightarrow\; -1 = 2B \;\rightarrow\; B = -1/2 \end{array}\right\}$$

Por tanto:

$$\int \frac{1}{1-x^2}\,dx = \int \left(\frac{1/2}{x+1} + \frac{-1/2}{x-1}\right) dx =$$

$$= \frac{1}{2}\,ln|x+1| + \frac{1}{2}\,ln|x-1| + k = ln \qquad\qquad + k$$

h) $\int \dfrac{(1-x)^2}{1+x}\,dx = \int \dfrac{x^2-2x+1}{x+1}\,dx = \int \left(x - 3 + \dfrac{4}{x+1}\right) dx =$

$$= \frac{x^2}{2} - 3x + 4\,ln|x+1| + k$$

24 **S** **Resuelve:**

a) $\displaystyle\int \frac{1}{1 + e^x}\, dx$

☛ ***En el numerador, suma y resta*** e^x.

b) $\displaystyle\int \frac{x + 3}{\sqrt{9 - x^2}}\, dx$

☛ ***Descomponla en suma de otras dos.***

a) $\displaystyle\int \frac{1}{1 + e^x}\, dx = \int \frac{1 + e^x - e^x}{1 + e^x}\, dx = \int \left(1 - \frac{e^x}{1 + e^x}\right) = x - \ln(1 + e^x) + k$

b) $\displaystyle\int \frac{x + 3}{\sqrt{9 - x^2}}\, dx = -\int \frac{-x}{\sqrt{9 - x^2}}\, dx + \int \frac{3}{\sqrt{9 - x^2}}\, dx =$

$$= -\sqrt{9 - x^2} + 3\int \frac{1/3}{\sqrt{1 - (x/3)^2}}\, dx = -\sqrt{9 - x^2} + 3 \text{ arc sen}\left(\frac{x}{3}\right) + k$$

25 **Resuelve por sustitución:**

a) $\displaystyle\int x\sqrt{x + 1}\, dx$

b) $\displaystyle\int \frac{dx}{x - \sqrt[4]{x}}$

c) $\displaystyle\int \frac{x}{\sqrt{x + 1}}\, dx$

d) $\displaystyle\int \frac{1}{\quad}\, dx$

e) $\displaystyle\int \frac{1}{x + \sqrt{x}}\, dx$

f) $\displaystyle\int \frac{\sqrt{x}}{1 + x}\, dx$

☛ ***a) Haz*** $x + 1 = t^2$. ***b) Haz*** $x = t^4$.

a) $\displaystyle\int x\sqrt{x + 1}\, dx$

Cambio: $x + 2 = t^2 \rightarrow dx = 2t\, dt$

$$\int x\sqrt{x + 1}\, dx = \int (t^2 - 1)\, t \cdot 2t\, dt = \int (2t^4 - 2t^2)\, dt = \frac{2t^5}{5} - \frac{2t^3}{3} + k =$$

$$= \frac{2\sqrt{(x + 1)^5}}{5} - \frac{2\sqrt{(x + 1)^3}}{3} + k$$

b) $\displaystyle\int \frac{dx}{x - \sqrt[4]{x}}$

Cambio: $x = t^4 \rightarrow dx = 4t^3\, dt$

$$\int \frac{dx}{x - \sqrt[4]{x}} = \int \frac{4t^3\, dt}{t^4 - t} = \int \frac{4t^2\, dt}{t^3 - 1} = \frac{4}{3}\int \frac{3t^2\, dt}{t^3 - 1} = \frac{4}{3} \ln|t^3 - 1| + k =$$

$$= \frac{4}{3} \ln\left|\sqrt[4]{x^3} - 1\right| + k$$

c) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

Cambio: $x + 1 = t^2 \;\rightarrow\; dx = 2t\,dt$

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{(t^2-1)}{t} \cdot 2t\,dt = \int (2t^2 - 2)\,dt = \frac{2t^3}{3} - 2t + k =$$

$$= \frac{2\sqrt{(x+1)^3}}{3} - 2\sqrt{x+1} + k$$

d) $\displaystyle\int \frac{1}{}\,dx$

Cambio: $x + 1 = t^2 \;\rightarrow\; dx = 2t\,dt$

$$\int \frac{1}{}\,dx = \int \frac{2t\,dt}{(t^2-1)\,t} = \int \frac{2\,dt}{(t+1)(t-1)}$$

Descomponemos en fracciones simples:

$$\frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} = \frac{A(t-1) + B(t+1)}{(t+1)(t-1)}$$

$2 = A(t-1) + B(t+1)$

Hallamos A y B:

$$\left.\begin{array}{lll} t = -1 \;\rightarrow\; 2 = -2A \;\rightarrow\; A = -1 \\ t = 1 \;\rightarrow\; 2 = 2B \;\rightarrow\; B = 1 \end{array}\right\}$$

Por tanto:

$$\int \frac{2\,dt}{(t+1)(t-1)} = \int \left(\frac{-1}{t+1} + \frac{1}{t-1} \right) dt = -ln\,|t+1| + ln\,|t-1| + k =$$

$$= ln\left|\frac{t-1}{t+1}\right| + k$$

Así:

$$\int \frac{1}{}\,dx = ln\left|\frac{\sqrt{x+1} - 1}{}\right| + k$$

e) $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$

Cambio: $x = t^2 \;\rightarrow\; dx = 2t\,dt$

$$\int \frac{1}{x+\sqrt{x}}\,dx = \int \frac{2t\,dt}{t^2+t} = \int \frac{2\,dt}{t+1} = 2\,ln\,|t+1| + k =$$

$$= 2\,ln\,(\sqrt{x} + 1) + k$$

f) $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$

Cambio: $x = t^2 \;\rightarrow\; dx = 2t\,dt$

$$\int \frac{\sqrt{x}}{1+x}\,dx = \int \frac{t \cdot 2t\,dt}{1+t^2} = \int \frac{2t^2\,dt}{1+t^2} = \int \left(2 - \frac{2}{1+t^2}\right) dt =$$

$$= 2t - 2\,arc\,tg\,t + k = 2\sqrt{x} - 2\,arc\,tg\,\sqrt{x} + k$$

26 **Resuelve, utilizando un cambio de variable, estas integrales:**

a) $\displaystyle\int \sqrt{9-4x^2}\,dx$ **b)** $\displaystyle\int \frac{dx}{e^{2x}-3e^x}$ **c)** $\displaystyle\int \frac{e^{3x}-e^x}{e^{2x}+1}\,dx$ **d)** $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

☛ ***a) Haz*** $sen\,t = 2x/3$.

a) $\displaystyle\int \sqrt{9-4x^2}\,dx$

Cambio: $sen\,t = \frac{2x}{3} \;\rightarrow\; x = \frac{3}{2}\,sen\,t \;\rightarrow\; dx = \frac{3}{2}\cos t\,dt$

$$\int \sqrt{9-4x^2}\,dx = \int \sqrt{9 - 4 \cdot \frac{9}{4}\,sen^2\,t} \cdot \frac{3}{2}\cos t\,dt = \int 3\cos t \cdot \frac{3}{2}\cos t\,dt =$$

$$= \frac{9}{2}\int \cos^2 t\,dt = \frac{9}{2}\int \left(\frac{1}{2} - \frac{\cos 2t}{2}\right) dt = \frac{9}{2}\left(\frac{1}{2}t + \frac{1}{4}\,sen\,2t\right) + k =$$

$$= \frac{9}{4}t + \frac{9}{8}\,sen\,2t + k = \frac{9}{4}\,arc\,sen\left(\frac{2x}{3}\right) + \frac{9}{8} \cdot 2\,sen\,t\cos t + k =$$

$$= \frac{9}{4}\,arc\,sen\left(\frac{2x}{3}\right) + \frac{9}{4} \cdot \frac{2x}{3}\sqrt{1 - \frac{4x^2}{9}} + k =$$

$$= \frac{9}{4}\,arc\,sen\left(\frac{2x}{3}\right) + \frac{x}{2} \cdot \sqrt{9-4x^2} + k$$

b) $\displaystyle\int \frac{dx}{e^{2x}-3e^x}$

Cambio: $e^x = t \;\rightarrow\; x = \ln t \;\rightarrow\; dx = \frac{1}{t}\,dt$

$$\int \frac{dx}{e^{2x}-3e^x} = \int \frac{1/t}{t^2-3t}\,dt = \int \frac{1}{t^3-3t^2}\,dt = \int \frac{1}{t^2(t-3)}\,dt$$

Descomponemos en fracciones simples:

$$\frac{1}{t^2(t-3)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-3} = \frac{At(t-3) + B(t-3) + Ct^2}{t^2(t-3)}$$

$$1 = At(t-3) + B(t-3) + Ct^2$$

Hallamos A, B y C:

$$\left.\begin{array}{llllll} t = 0 & \to & 1 = -3B & \to & B = -1/3 \\ t = 3 & \to & 1 = 9C & \to & C = 1/9 \\ t = 1 & \to & 1 = -2A - 2B + C & \to & A = -1/9 \end{array}\right\}$$

Así, tenemos que:

$$\int \frac{1}{t^2(t-3)}\,dt = \int \left(\frac{-1/9}{t} + \frac{-1/3}{t^2} + \frac{1/9}{t-3} \right) dt =$$

$$= \frac{-1}{9}\, ln\,|t| + \frac{1}{3t} + \frac{1}{9}\, ln\,|t-3| + k$$

Por tanto:

$$\int \frac{dx}{e^{2x} - 3e^x} = \frac{-1}{9}\, ln\, e^x + \frac{1}{3e^x} + \frac{1}{9}\, ln\,|e^x - 3| + k =$$

$$= -\frac{1}{9}\,x + \frac{1}{3e^x} + \frac{1}{9}\, ln\,|e^x - 3| + k$$

c) $\displaystyle\int \frac{e^{3x} - e^x}{e^{2x} + 1}\,dx$

Cambio: $e^x = t \;\to\; x = ln\, t \;\to\; dx = \dfrac{1}{t}\,dt$

$$\int \frac{e^{3x} - e^x}{e^{2x} + 1}\,dx = \int \frac{t^3 - t}{t^2 + 1} \cdot \frac{1}{t}\,dt = \int \frac{t^2 - 1}{t^2 + 1}\,dt = \int \left(1 - \frac{2}{t^2 + 1}\right) dt =$$

$$= t - 2\, arc\, tg\, t + k = e^x - 2\, arc\, tg\,(e^x) + k$$

d) $\displaystyle\int \frac{1}{1 + \sqrt{x}}\,dx$

Cambio: $x = t^2 \;\to\; dx = 2t\,dt$

$$\int \frac{1}{1 + \sqrt{x}}\,dx = \int \frac{2t\,dt}{1 + t} = \int \left(2 - \frac{2}{1 + t}\right) dt = 2t - 2\, ln\,|1 + t| + k =$$

$$= 2\sqrt{x} - 2\, ln\,(1 + \sqrt{x}) + k$$

27 S **Encuentra la primitiva de $f(x) = \dfrac{1}{1 + 3x}$ que se anula para $x = 0$.**

$$F(x) = \int \frac{1}{1 + 3x}\,dx = \frac{1}{3}\int \frac{3}{1 + 3x}\,dx = \frac{1}{3}\, ln\,|1 + 3x| + k$$

$$F(0) = k = 0$$

Por tanto: $F(x) = \dfrac{-1}{3}\, ln\,|1 + 3x|$

28 **Halla la función F para la que $F'(x) = \dfrac{1}{x^2}$ y $F(1) = 2$.**

$$F(x) = \int \frac{1}{x^2}\, dx = \frac{-1}{x} + k$$

$$F(1) = -1 + k = 2 \quad \Rightarrow \quad k = 3$$

Por tanto: $F(x) = \dfrac{-1}{x} + 3$

29 **De todas las primitivas de la función $y = 4x - 6$, ¿cuál de ellas toma el valor 4 para $x = 1$?**

$$F(x) = \int (4x - 6)\, dx = 2x^2 - 6x + k$$

$$F(1) = 2 - 6 + k = 4 \quad \Rightarrow \quad k = 8$$

Por tanto: $F(x) = 2x^2 - 6x + 8$

30 **Halla $f(x)$ sabiendo que $f''(x) = 6x$, $f'(0) = 1$ y $f(2) = 5$.**

$$\left.\begin{array}{l} f'(x) = \displaystyle\int 6x\, dx = 3x^2 + c \\ f'(0) = c = 1 \end{array}\right\} \quad f'(x) = 3x^2 + 1$$

$$\left.\begin{array}{l} f(x) = \displaystyle\int (3x^2 + 1)\, dx = x^3 + x + k \\ \\ \end{array}\right\}$$

Por tanto: $f(x) = x^3 + x - 5$

31 **Resuelve las siguientes integrales por sustitución:**

a) $\displaystyle\int \frac{e^x}{1 - \sqrt{e^x}}\, dx$ **b)** $\displaystyle\int \sqrt{e^x - 1}\, dx$

a) Haz $\sqrt{e^x} = t$. b) Haz $\sqrt{e^x - 1} = t$.

a) $\displaystyle\int \frac{e^x}{1 - \sqrt{e^x}}\, dx$

Cambio: $\sqrt{e^x} = t \;\rightarrow\; e^{x/2} = t \;\rightarrow\; \dfrac{x}{2} = \ln t \;\rightarrow\; dx = \dfrac{2}{t}\, dt$

$$\int \frac{e^x}{1 - \sqrt{e^x}} = \int \frac{t^2 \cdot (2/t)\, dt}{1 - t} = \int \frac{2t\, dt}{1 - t} = \int \left(-2 + \frac{2}{1 - t}\right) dt =$$

$$= -2t - 2 \ln|1 - t| + k = -2\sqrt{e^x} - 2 \ln|1 - \sqrt{e^x}| + k$$

b) $\int \sqrt{e^x - 1}\ dx$

Cambio: $\sqrt{e^x - 1} = t \ \rightarrow\ e^x = t^2 + 1 \ \rightarrow\ x = ln\,(t^2 + 1) \ \rightarrow\ dx = \frac{2t}{t^2 + 1}\,dt$

$$\int \sqrt{e^x - 1}\ dx = \int t \cdot \frac{2t}{t^2 + 1}\,dt = \int \frac{2t^2}{t^2 + 1}\,dt = \int \left(2 - \frac{2}{t^2 + 1}\right) dt =$$

$$= 2t - 2\ arc\ tg\ t + k = 2\sqrt{e^x - 1} - 2\ arc\ tg\ \sqrt{e^x - 1} + k$$

32 **Calcula** $\int \frac{sen^2\,x}{1 + cos\,x}\,dx$.

☛ ***Multiplica numerador y denominador por*** $1 - cos\,x$.

$$\int \frac{sen^2\,x}{1 + cos\,x}\,dx = \int \frac{sen^2\,x\,(1 - cos\,x)}{(1 + cos\,x)(1 - cos\,x)}\,dx = \int \frac{sen^2\,x\,(1 - cos\,x)}{1 - cos^2\,x}\,dx =$$

$$= \int \frac{sen^2\,x\,(1 - cos\,x)}{sen^2\,x}\,dx = \int (1 - cos\,x)\,dx = x - sen\,x + k$$

Página 352

33 **S** **Encuentra una primitiva de la función:**

$$f(x) = x^2\,sen\,x$$

cuyo valor para $x = \pi$ **sea 4.**

$$F(x) = \int x^2\,sen\,x\,dx$$

Integramos por partes:

$$\begin{cases} u = x^2 \ \rightarrow\ du = 2x\,dx \\ dv = sen\,x\,dx \ \rightarrow\ v = -cos\,x \end{cases}$$

$$F(x) = -x^2\,cos\,x + 2\underbrace{\int x\,cos\,x\,dx}_{I_1}$$

$$\begin{cases} u_1 = x \ \rightarrow\ du_1 = dx \\ dv_1 = cos\,x\,dx \ \rightarrow\ v_1 = sen\,x \end{cases}$$

$$I_1 = x\,sen\,x - \int sen\,x\,dx = x\,sen\,x + cos\,x$$

Por tanto:

$$\left.\begin{array}{l} F(x) = -x^2\,cos\,x + 2\ x\,sen\,x + 2\,cos\,x + k \\ F(\pi) = \pi^2 - 2 + k = 4 \ \Rightarrow\ k = 6 - \pi^2 \end{array}\right\}$$

$$F(x) = -x^2\,cos\,x + 2\ x\,sen\,x + 2\,cos\,x + 6 - \pi^2$$

34 **S** **Determina la función $f(x)$ sabiendo que:**

$$f''(x) = x \ln x, \; f'(1) = 0 \quad \text{y} \quad f(e) = \frac{e}{4}$$

$$f'(x) = \int x \ln x \, dx$$

Integramos por partes:

$$\begin{cases} u = \ln x \;\rightarrow\; du = \dfrac{1}{x}\,dx \\ dv = x\,dx \;\rightarrow\; v = \dfrac{x^2}{2} \end{cases}$$

$$\left. \begin{aligned} & f'(x) = \frac{x^2}{2} \ln x - \int \frac{x}{2}\,dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + k = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + k \\ & f'(1) = \frac{1}{2}\left(-\frac{1}{2}\right) + k = -\frac{1}{4} + k = 0 \;\Rightarrow\; k = \frac{1}{4} \end{aligned} \right\}$$

$$f'(x) = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + \frac{1}{4}$$

$$f(x) = \int \left[\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + \frac{1}{4}\right] dx = \underbrace{\int \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) dx}_{I} + \frac{1}{4}x$$

$$\begin{cases} u = \left(\ln x - \dfrac{1}{2}\right) \;\rightarrow\; du = \dfrac{1}{x}\,dx \\ dv = \dfrac{x^2}{2}\,dx \;\rightarrow\; v = \dfrac{x^3}{6} \end{cases}$$

$$I = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \int \frac{x^2}{6}\,dx = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \frac{x^3}{18} + k$$

Por tanto:

$$\left. \begin{aligned} & f(x) = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \frac{x^3}{18} + \frac{1}{4}x + k \\ & f(e) = \frac{e^3}{12} - \frac{e^3}{18} + \frac{e}{4} + k = \frac{e^3}{36} + \frac{e}{4} + k = \frac{e}{4} \;\Rightarrow\; k = -\frac{e^3}{36} \end{aligned} \right\}$$

$$f(x) = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \frac{x^3}{18} + \frac{1}{4}x - \frac{e^3}{36}$$

35 **S** **Calcula la expresión de una función $f(x)$ tal que $f'(x) = x\, e^{-x^2}$ y que $f(0) = \frac{1}{2}$.**

$$f(x) = \int x\, e^{-x^2}\, dx = -\frac{1}{2}\int -2x\, e^{-x^2}\, dx = -\frac{1}{2}\, e^{-x^2} + k$$

$$f(0) = -\frac{1}{2} + k = \frac{1}{2} \quad \Rightarrow \quad k = 1$$

Por tanto: $f(x) = -\frac{1}{2}\, e^{-x^2} + 1$

36 **S** **Encuentra la función derivable $f: [-1, 1] \to \mathbb{R}$ que cumple $f(1) = -1$ y**

$$f'(x) = \begin{cases} x^2 - 2x & \text{si } -1 \le x < 0 \\ e^x - 1 & \text{si } 0 \le x \le 1 \end{cases}$$

- Si $x \ne 0$:

$$f(x) = \begin{cases} \dfrac{x^3}{3} - x^2 + k & \text{si } -1 \le x < 0 \\ e^x - x + c & \text{si } 0 < x \le 1 \end{cases}$$

- Hallamos k y c teniendo en cuenta que $f(1) = -1$ y que $f(x)$ ha de ser continua en $x = 0$.

$$f(1) = -1 \quad \Rightarrow \quad e - 1 + c = -1 \quad \Rightarrow \quad c = -e$$

$$\left.\begin{array}{l} \lim\limits_{x \to 0^-} f(x) = k \\ \lim\limits_{x \to 0^+} f(x) = 1 - e \end{array}\right\} \; k = 1 - e$$

Por tanto: $f(x) = \begin{cases} \dfrac{x^3}{3} - x^2 + 1 - e & \text{si } -1 \le x < 0 \\ e^x - x - e & \text{si } 0 \le x \le 1 \end{cases}$

37 **S** **De una función derivable se sabe que pasa por el punto $A(-1, -4)$ y que su derivada es:**

$$f'(x) = \begin{cases} 2 - x & \text{si } x \le 1 \\ 1/x & \text{si } x > 1 \end{cases}$$

a) Halla la expresión de $f(x)$.

b) Obtén la ecuación de la recta tangente a $f(x)$ en $x = 2$.

a) Si $x \ne 1$:

$$f(x) = \begin{cases} 2x - \dfrac{x^2}{2} + k & \text{si } x < 1 \\ \ln x + c & \text{si } x > 1 \end{cases}$$

Hallamos k y c teniendo en cuenta que $f(-1) = -4$ y que $f(x)$ ha de ser continua en $x = 1$.

$$f(-1) = -\frac{5}{2} + k = -4 \quad \Rightarrow \quad k = -\frac{3}{2}$$

$$\left.\begin{array}{l} \lim\limits_{x \to 1^-} f(x) = \frac{3}{2} - \frac{3}{2} = 0 \\ \lim\limits_{x \to 1^+} f(x) = c \end{array}\right\} c = 0$$

$$\text{Por tanto: } f(x) = \begin{cases} 2x - \frac{x^2}{2} - \frac{3}{2} & \text{si } x < 1 \\ \ln x & \text{si } x \geq 1 \end{cases}$$

b) $f(2) = \ln 2; \quad f'(2) = \frac{1}{2}$

La ecuación de la recta tangente será: $y = \ln 2 + \frac{1}{2}(x - 2)$

38 **S** **Calcula:**

a) $\int |1 - x|\, dx$ **b)** $\int (3 + |x|)\, dx$ **c)** $\int |2x - 1|\, dx$ **d)** $\int \left|\frac{x}{2} - 2\right| dx$

a) $\int |1 - x|\, dx$

$$|1 - x| = \begin{cases} 1 - x & \text{si } x < 1 \\ -1 + x & \text{si } x \geq 1 \end{cases}$$

$$f(x) = \int |1 - x|\, dx = \begin{cases} x - \frac{x^2}{2} + k & \text{si } x < 1 \\ -x + \frac{x^2}{2} + c & \text{si } x \geq 1 \end{cases}$$

En $x = 1$, la función ha de ser continua:

$$\left.\begin{array}{l} \lim\limits_{x \to 1^-} f(x) = \frac{1}{2} + k \\ \lim\limits_{x \to 1^+} f(x) = -\frac{1}{2} + c \end{array}\right\} \frac{1}{2} + k = -\frac{1}{2} + c \quad \Rightarrow \quad c = 1 + k$$

Por tanto:

$$\int |1 - x|\, dx = \begin{cases} x - \frac{x^2}{2} + k & \text{si } x < 1 \\ -x + \frac{x^2}{2} + 1 + k & \text{si } x \geq 1 \end{cases}$$

b) $\int (3 + |x|)\,dx$

$$3 + |x| = \begin{cases} 3 - x & \text{si } x < 0 \\ 3 + x & \text{si } x \geq 0 \end{cases}$$

$$f(x) = \int (3 + |x|)\,dx = \begin{cases} 3x - \dfrac{x^2}{2} + k & \text{si } x < 0 \\ 3x + \dfrac{x^2}{2} + c & \text{si } x \geq 0 \end{cases}$$

En $x = 0$, $f(x)$ ha de ser continua:

$$\left.\begin{array}{l} \lim\limits_{x \to 0^-} f(x) = k \\ \lim\limits_{x \to 0^+} f(x) = c \end{array}\right\} c = k$$

Por tanto:

$$\int (3 + |x|)\,dx = \begin{cases} 3x - \dfrac{x^2}{2} + k & \text{si } x < 0 \\ 3x + \dfrac{x^2}{2} + k & \text{si } x \geq 0 \end{cases}$$

c) $\int |2x - 1|\,dx$

$$|2x - 1| = \begin{cases} -2x + 1 & \text{si } x < 1/2 \\ 2x - 1 & \text{si } x \geq 1/2 \end{cases}$$

$$f(x) = \int |2x - 1|\,dx = \begin{cases} -x^2 + x + k & \text{si } x < \dfrac{1}{2} \\ x^2 - x + c & \text{si } x \geq \dfrac{1}{2} \end{cases}$$

$f(x)$ ha de ser continua en $x = \dfrac{1}{2}$:

$$\left.\begin{array}{l} \lim\limits_{x \to (1/2)^-} f(x) = \dfrac{1}{4} + k \\ \lim\limits_{x \to (1/2)^+} f(x) = -\dfrac{1}{4} + c \end{array}\right\} \dfrac{1}{4} + k = -\dfrac{1}{4} + c \Rightarrow c = \dfrac{1}{2} + k$$

Por tanto:

$$\int |2x - 1|\,dx = \begin{cases} -x^2 + x + k & \text{si } x < \dfrac{1}{2} \\ x^2 - x + \dfrac{1}{2} + k & \text{si } x \geq \dfrac{1}{2} \end{cases}$$

d) $\int \left| \frac{x}{2} - 2 \right| dx$

$$\left| \frac{x}{2} - 2 \right| = \begin{cases} -\frac{x}{2} + 2 & \text{si } x < 4 \\ \frac{x}{2} - 2 & \text{si } x \geq 4 \end{cases}$$

$$f(x) = \int \left| \frac{x}{2} - 2 \right| dx = \begin{cases} -\frac{x^2}{4} + 2x + k & \text{si } x < 4 \\ \frac{x^2}{4} - 2x + c & \text{si } x \geq 4 \end{cases}$$

$f(x)$ ha de ser continua en $x = 4$:

$$\left. \begin{array}{l} \lim\limits_{x \to 4^-} f(x) = 4 + k \\ \lim\limits_{x \to 4^+} f(x) = -4 + c \end{array} \right\} \; 4 + k = -4 + c \;\; \Rightarrow \;\; c = 8 + k$$

Por tanto:

$$\int \left| \frac{x}{2} - 2 \right| dx = \begin{cases} -\frac{x^2}{4} + 2x + k & \text{si } x < 4 \\ \frac{x^2}{4} - 2x + 8 + k & \text{si } x \geq 4 \end{cases}$$

39 **Calcula** $\int \frac{1}{sen^2\, x \cos^2 x} dx$**.**

$$\int \frac{1}{sen^2\, x \cos^2 x} dx = \int \frac{sen^2\, x + \cos^2 x}{sen^2\, x \cos^2 x} dx =$$

$$= \int \frac{sen^2\, x}{sen^2\, x \cos^2 x} dx + \int \frac{\cos^2 x}{sen^2\, x \cos^2 x} dx =$$

$$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{sen^2\, x} dx = tg\, x - cotg\, x + k$$

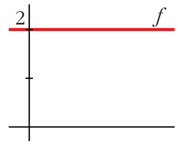
CUESTIONES TEÓRICAS

40 **Prueba que, si $F(x)$ es una primitiva de $f(x)$ y C un número real cualquiera, la función $F(x) + C$ es también una primitiva de $f(x)$.**
S

$F(x)$ primitiva de $f(x) \Leftrightarrow F'(x) = f(x)$

$$\big(F(x) + C\big)' = F'(x) = f(x) \;\; \Rightarrow \;\; F(x) + C \text{ es primitiva de } f(x).$$

41 **Representa tres primitivas de la función f cuya gráfica es esta:**



$$f(x) = 2 \Rightarrow F(x) = 2x + k$$

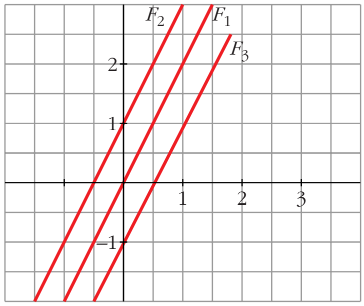
Por ejemplo:

$F_1(x) = 2x$

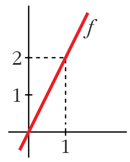
$F_2(x) = 2x + 1$

$F_3(x) = 2x - 1$

cuyas gráficas son:



42 **Representa tres primitivas de la función f:**



$$f(x) = 2x \Rightarrow F(x) = x^2 + k$$

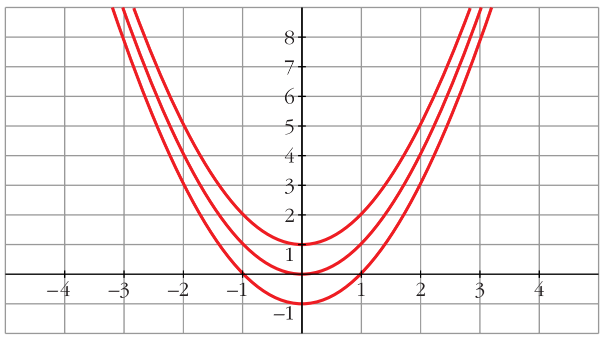
Por ejemplo:

$F_1(x) = x^2$

$F_2(x) = x^2 + 1$

$F_3(x) = x^2 - 1$

cuyas gráficas son:



43 **Sabes que una primitiva de la función $f(x) = \frac{1}{x}$ es $F(x) = ln\,|x|$. ¿Por qué se toma el valor absoluto de x?**

$f(x) = \frac{1}{x}$ está definida para todo $x \neq 0$; y es la derivada de la función:

$$F(x) = \begin{cases} ln\,x & \text{si } x > 0 \\ ln\,(-x) & \text{si } x < 0 \end{cases}$$

es decir, de $F(x) = ln\,|x|$.

44 **En una integral hacemos el cambio de variable $e^x = t$. ¿Cuál es la expresión de dx en función de t?**

$$e^x = t \;\rightarrow\; x = ln\,t \;\rightarrow\; dx = \frac{1}{t}\,dt$$

45 **Comprueba que:** $\displaystyle\int \frac{1}{\cos x}\,dx = \ln|\sec x + \operatorname{tg} x| + k$

Tenemos que probar que la derivada de $f(x) = \ln|\sec x + \operatorname{tg} x| + k$ es $f'(x) = \dfrac{1}{\cos x}$.

Derivamos $f(x) = \ln\left|\dfrac{1 + \operatorname{sen} x}{\cos x}\right| + k$:

$$f'(x) = \frac{\dfrac{\cos^2 x + \operatorname{sen} x(1 + \operatorname{sen} x)}{\cos^2 x}}{\dfrac{1 + \operatorname{sen} x}{\cos x}} = \frac{\dfrac{\cos^2 x + \operatorname{sen} x + \operatorname{sen}^2 x}{\cos x}}{1 + \operatorname{sen} x} =$$

$$= \frac{1 + \operatorname{sen} x}{(1 + \operatorname{sen} x)\cos x} = \frac{1}{\cos x}$$

46 **Comprueba que:** $\displaystyle\int \frac{1}{\operatorname{sen} x \cos x}\,dx = \ln|\operatorname{tg} x| + k$

Tenemos que comprobar que la derivada de la función $f(x) = \ln|\operatorname{tg} x| + k$ es $f'(x) = \dfrac{1}{\operatorname{sen} x \cos x}$.

Derivamos $f(x)$:

$$f'(x) = \frac{1/\cos^2 x}{\operatorname{tg} x} = \frac{1/\cos^2 x}{\operatorname{sen} x/\cos x} = \frac{1}{\operatorname{sen} x \cos x}$$

47 **Sin utilizar cálculo de derivadas, prueba que:**

$$F(x) = \frac{1}{1 + x^4} \quad \text{y} \quad G(x) = \frac{-x^4}{1 + x^4}$$

son dos primitivas de una misma función.

Si $F(x)$ y $G(x)$ son dos primitivas de una misma función, su diferencia es una constante. Veámoslo:

$$F(x) - G(x) = \frac{1}{1 + x^4} - \left(\frac{-x^4}{1 + x^4}\right) = \frac{1 + x^4}{1 + x^4} = 1$$

Por tanto, hemos obtenido que: $F(x) = G(x) + 1$

Luego las dos son primitivas de una misma función.

48 **Sean f y g dos funciones continuas y derivables que se diferencian en una constante. ¿Podemos asegurar que f y g tienen una misma primitiva?**

No. Por ejemplo:

$$\left.\begin{array}{l} f(x) = 2x + 1 \;\rightarrow\; F(x) = x^2 + x + k \\ g(x) = 2x + 2 \;\rightarrow\; G(x) = x^2 + 2x + c \end{array}\right\}$$

$f(x)$ y $g(x)$ son continuas, derivables y se diferencian en una constante (pues $f(x) = g(x) - 1$).

Sin embargo, sus primitivas, $F(x)$ y $G(x)$ respectivamente, son distintas, cualesquiera que sean los valores de k y c.

Página 353

PARA PROFUNDIZAR

49 Para integrar una función cuyo denominador es un polinomio de segundo grado sin raíces reales, distinguiremos dos casos:

a) Si el numerador es constante, transformamos el denominador para obtener un binomio al cuadrado. La solución será un arco tangente:

$$\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x+2)^2 + 1}$$

(Completa la resolución).

b) Si el numerador es de primer grado, se descompone en un logaritmo neperiano y un arco tangente:

$$\int \frac{(x+5)\,dx}{x^2 + 2x + 3} = \frac{1}{2}\int \frac{2x + 10}{x^2 + 2x + 3}\,dx = \frac{1}{2}\int \frac{2x + 2}{x^2 + 2x + 3}\,dx + \frac{1}{2}\int \frac{8\,dx}{x^2 + 2x + 3}$$

(Completa su resolución).

a) $\displaystyle\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x+2)^2 + 1} = arc\ tg(x+2) + k$

b) $\displaystyle\int \frac{(x+5)\,dx}{x^2 + 2x + 3} = \frac{1}{2}\int \frac{2x + 10}{x^2 + 2x + 3}\,dx = \frac{1}{2}\int \frac{2x + 2}{x^2 + 2x + 3}\,dx + \frac{1}{2}\int \frac{8\,dx}{x^2 + 2x + 3} =$

$$= \frac{1}{2}\ln(x^2 + 2x + 3) + 4\int \frac{dx}{(x+1)^2 + 2} =$$

$$= \frac{1}{2}\ln(x^2 + 2x + 3) + 2\int \frac{dx}{\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1} =$$

$$= \frac{1}{2}\ln(x^2 + 2x + 3) + 2\sqrt{2}\int \frac{(1/\sqrt{2})\,dx}{\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1} =$$

$$= \frac{1}{2}\ln(x^2 + 2x + 3) + 2\sqrt{2}\ arc\ tg\left(\frac{x+1}{\sqrt{2}}\right) + k$$

50 **Observa cómo se resuelve esta integral:**

$$I = \int \frac{x + 1}{x^3 + 2x^2 + 3x}\, dx$$

$$x^3 + 2x^2 + 3x = x(x^2 + 2x + 3)$$

La fracción se descompone así: $\dfrac{x + 1}{x^3 + 2x^2 + 3x} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 2x + 3}$

Obtenemos: $A = \dfrac{1}{3},\ B = -\dfrac{1}{3},\ C = \dfrac{1}{3}$

Sustituimos: $I = \dfrac{1}{3}\displaystyle\int \frac{1}{x}\, dx - \frac{1}{3}\int \frac{x - 1}{x^2 + 2x + 3}\, dx$

(Completa su resolución).

Completamos la resolución:

$$I = \frac{1}{3}\int \frac{1}{x}\, dx - \frac{1}{3}\int \frac{x - 1}{x^2 + 2x + 3}\, dx =$$

$$= \frac{1}{3} ln|x| - \frac{1}{6}\int \frac{2x - 2}{x^2 + 2x + 3}\, dx = \frac{1}{3} ln|x| - \frac{1}{6}\int \frac{2x + 2 - 4}{x^2 + 2x + 3}\, dx =$$

$$= \frac{1}{3} ln|x| - \frac{1}{6}\int \frac{2x - 2}{x^2 + 2x + 3}\, dx + \frac{2}{3}\int \frac{dx}{x^2 + 2x + 3} \overset{(*)}{=}$$

$$= \frac{1}{3} ln|x| - \frac{1}{6} ln(x^2 + 2x + 3) + \frac{\sqrt{2}}{3}\, arc\ tg\left(\frac{x + 1}{\sqrt{2}}\right) + k$$

(*) $\left(\text{Ver en el ejercicio 49 apartado b) el cálculo de } \displaystyle\int \frac{dx}{x^2 + 2x + 3}\right)$.

51 **Resuelve las siguientes integrales:**

a) $\displaystyle\int \frac{2x - 1}{x^3 + x}\, dx$ **b)** $\displaystyle\int \frac{1}{x^3 + 1}\, dx$ **c)** $\displaystyle\int \frac{x^2 + 3x + 8}{x^2 + 9}\, dx$

d) $\displaystyle\int \frac{2x + 10}{x^2 + x + 1}\, dx$ **e)** $\displaystyle\int \frac{2}{x^2 + 3x + 4}\, dx$ **f)** $\displaystyle\int \frac{dx}{(x + 1)^2 (x^2 + 1)}$

e) Multiplica numerador y denominador por 4.

a) $\displaystyle\int \frac{2x - 1}{x^3 + x}\, dx = \int \frac{2x - 1}{x(x^2 + 1)}\, dx$

Descomponemos la fracción:

$$\frac{2x - 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{A(x^2 + 1) + Bx^2 + Cx}{x(x^2 + 1)}$$

$$2x - 1 = A(x^2 + 1) + Bx^2 + Cx$$

Hallamos A, B y C:

$$\left.\begin{array}{llll} x = 0 & \rightarrow & -1 = A & \\ x = 1 & \rightarrow & 1 = 2A + B + C & \rightarrow \; 3 = B + C \\ x = -1 & \rightarrow & -3 = 2A + B - C & \rightarrow \; -1 = B - C \end{array}\right\} \begin{array}{l} A = -1 \\ B = 1 \\ C = 2 \end{array}$$

Por tanto:

$$\int \frac{2x-1}{x^3+x}\,dx = \int \left(\frac{-1}{x} + \frac{x+2}{x^2+1} \right) dx =$$

$$= \int \frac{-1}{x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + 2\int \frac{dx}{x^2+1} =$$

$$= -ln|x| + \frac{1}{2}\,ln(x^2+1) + 2\,arc\,tg\,x + k$$

b) $\displaystyle\int \frac{1}{x^3+1}\,dx = \int \frac{dx}{(x+1)(x^2-x+1)}$

Descomponemos la fracción:

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} =$$

$$= \frac{A(x^2-x+1) + Bx(x+1) + C(x+1)}{(x+1)(x^2-x+1)}$$

$$1 = A(x^2-x+1) + Bx(x+1) + C(x+1)$$

Hallamos A, B y C:

$$\left.\begin{array}{lllll} x = -1 & \rightarrow & 1 = 3A & \rightarrow & A = 1/3 \\ x = 0 & \rightarrow & 1 = A + C & \rightarrow & C = 2/3 \\ x = 1 & \rightarrow & 1 = A + 2B + 2C & \rightarrow & B = -1/3 \end{array}\right\}$$

Por tanto:

$$\int \frac{1}{x^3+1}\,dx = \int \frac{-1/3}{x+1}\,dx + \int \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\int \frac{2x-4}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\int \frac{2x-1-3}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2-x+1} =$$

$$= \frac{1}{3} ln|x+1| - \frac{1}{6} ln(x^2 - x + 1) + \frac{1}{2} \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} =$$

$$= \frac{1}{3} ln|x+1| - \frac{1}{6} ln(x^2 - x + 1) + \frac{1}{2} \int \frac{4/3}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} dx =$$

$$= \frac{1}{3} ln|x+1| - \frac{1}{6} ln(x^2 - x + 1) + \frac{\sqrt{3}}{3} \int \frac{2/\sqrt{3}}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1} dx =$$

$$= \frac{1}{3} ln|x+1| - \frac{1}{6} ln(x^2 - x + 1) + \frac{\sqrt{3}}{3} \, arc\ tg \left(\frac{2x-1}{\sqrt{3}}\right) + k$$

c) $\displaystyle\int \frac{x^2 + 3x + 8}{x^2 + 9} dx = \int \left(1 + \frac{3x - 1}{x^2 + 9}\right) dx = x + \int \frac{3x}{x^2 + 9} dx - \int \frac{dx}{x^2 + 9} =$

$$= x + \frac{3}{2} \int \frac{2x}{x^2 + 9} dx - \int \frac{1/9}{(x/3)^2 + 1} dx =$$

$$= x + \frac{3}{2} ln(x^2 + 9) - \frac{1}{3} \, arc\ tg \left(\frac{x}{3}\right) + k$$

d) $\displaystyle\int \frac{2x + 10}{x^2 + x + 1} dx = \int \frac{2x + 1 + 9}{x^2 + x + 1} dx = \int \frac{2x + 1}{x^2 + x + 1} dx + 9 \int \frac{1}{x^2 + x + 1} dx =$

$$= ln(x^2 + x + 1) + 9 \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} =$$

$$= ln(x^2 + x + 1) + 6\sqrt{3} \int \frac{2/\sqrt{3}}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} dx =$$

$$= ln(x^2 + x + 1) + 6\sqrt{3} \, arc\ tg \left(\frac{2x+1}{\sqrt{3}}\right) + k$$

e) $\displaystyle\int \frac{2}{x^2 + 3x + 4} dx = \int \frac{8}{4x^2 + 12x + 16} dx = \int \frac{8}{(2x + 3)^2 + 7} dx =$

$$= \int \frac{8/7}{\left(\frac{2x+3}{\sqrt{7}}\right)^2 + 1} dx = \frac{8}{7} \cdot \frac{\sqrt{7}}{2} \int \frac{2/\sqrt{7}}{\left(\frac{2x+3}{\sqrt{7}}\right)^2 + 1} dx =$$

$$= \frac{4\sqrt{7}}{7} \, arc\ tg \left(\frac{2x+3}{\sqrt{7}}\right) + k$$

f) $\displaystyle\int \frac{dx}{(x+1)^2\,(x^2+1)}$

Descomponemos la fracción:

$$\frac{1}{(x+1)^2\,(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

$$1 = A(x+1)(x^2+1) + B(x^2+1) + Cx(x+1)^2 + D(x+1)^2$$

Hallamos A, B, C y D:

$$\left.\begin{array}{lll} x=-1 & \rightarrow & 1 = 2B \;\rightarrow\; B = 1/2 \\ x=0 & \rightarrow & 1 = A+B+D \\ x=1 & \rightarrow & 1 = 4A+2B+4C+4D \\ x=-2 & \rightarrow & 1 = -5A+5B-2C+D \end{array}\right\} \begin{array}{l} A = 1/2 \\ B = 1/2 \\ C = -1/2 \\ D = 0 \end{array}$$

Por tanto:

$$\int \frac{dx}{(x+1)^2\,(x^2+1)} = \int\left(\frac{1/2}{x+1} + \frac{1/2}{(x+1)^2} - \frac{1}{2}\cdot\frac{x}{x^2+1}\right) dx =$$

$$= \frac{1}{2}\,ln|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\,ln(x^2+1) + k$$

PARA PENSAR UN POCO MÁS

52 Se llama ecuación diferencial de primer orden a una ecuación en la que, además de las variables x e y, figura también y'. Resolver una ecuación diferencial es buscar una función $y = f(x)$ que verifique la ecuación propuesta.

Por ejemplo, la ecuación $x\,y^2 + y' = 0$ se resuelve así:

$$y' = -x\,y^2 \;\rightarrow\; \frac{dy}{dx} = -x\,y^2 \;\rightarrow\; dy = -x\,y^2\,dx$$

Separamos las variables:

$$\frac{dy}{y^2} = -x\,dx \;\rightarrow\; \int \frac{dy}{y^2} = \int (-x)\,dx$$

$$-\frac{1}{y} = -\frac{x^2}{2} + k \;\rightarrow\; y = \frac{2}{x^2 - 2k}$$

Hay infinitas soluciones.

Busca la solución que pasa por el punto (0, 2) y comprueba que la curva que obtienes verifica la ecuación propuesta.

- Buscamos la solución que pasa por el punto $(0, 2)$:

$$y = \frac{2}{x^2-2k} \;\rightarrow\; 2 = \frac{2}{-2k} \;\Rightarrow\; -4k = 2 \;\Rightarrow\; k = \frac{-1}{2}$$

Por tanto: $y = \dfrac{2}{x^2+1}$

• Comprobamos que verifica la ecuación $xy^2 + y' = 0$:

$$xy^2 + y' = x\left(\frac{2}{x^2+1}\right)^2 - \frac{4x}{(x^2+1)^2} = x \cdot \frac{4}{(x^2+1)^2} - \frac{4x}{(x^2+1)^2} =$$

$$= \frac{4x}{(x^2+1)^2} - \frac{4x}{(x^2+1)^2} = 0$$

53 **Resuelve las siguientes ecuaciones:**

a) $y\,y' - x = 0$ **b)** $y^2\,y' - x^2 = 1$ **c)** $y' - x\,y = 0$

d) $y'\sqrt{x} - y = 0$ **e)** $y'\,e^y + 1 = e^x$ **f)** $x^2\,y' + y^2 + 1 = 0$

a) $yy' - x = 0$

$$y' = \frac{x}{y} \quad\Rightarrow\quad \frac{dy}{dx} = \frac{x}{y} \quad\Rightarrow\quad y\,dy = x\,dx \quad\Rightarrow\quad \int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + k \quad\Rightarrow\quad y^2 = x^2 + 2k$$

b) $y^2\,y' - x^2 = 1$

$$y' = \frac{1+x^2}{y^2} \quad\Rightarrow\quad \frac{dy}{dx} = \frac{1+x^2}{y^2} \quad\Rightarrow\quad y^2\,dy = (1+x^2)\,dx$$

$$\int y^2\,dy = \int (1+x^2)\,dx \quad\Rightarrow\quad \frac{y^3}{3} = x + \frac{x^3}{3} + k \quad\Rightarrow$$

$$\Rightarrow\quad y^3 = 3x + x^3 + 3k \quad\Rightarrow\quad y = \sqrt[3]{3x + x^3 + 3k}$$

c) $y' - x\,y = 0$

$$y' = xy \quad\Rightarrow\quad \frac{dy}{dx} = xy \quad\Rightarrow\quad \frac{dy}{y} = x\,dx \quad\Rightarrow\quad \int \frac{dy}{y} = \int x\,dx$$

$$ln\,|y| = \frac{x^2}{2} + k \quad\Rightarrow\quad |y| = e^{(x^2/2)+k}$$

d) $y'\sqrt{x} - y = 0$

$$y' = \frac{y}{\sqrt{x}} \quad\Rightarrow\quad \frac{dy}{dx} = \frac{y}{\sqrt{x}} \quad\Rightarrow\quad \frac{dy}{y} = \frac{dx}{\sqrt{x}} \quad\Rightarrow\quad \int \frac{dy}{y} = \int \frac{dx}{\sqrt{x}}$$

$$ln\,|y| = 2\sqrt{x} + k \quad\Rightarrow\quad |y| = e^{2\sqrt{x}+k}$$

e) $y'\,e^y + 1 = e^x$

$$y' = \frac{e^x - 1}{e^y} \quad\Rightarrow\quad \frac{dy}{dx} = \frac{e^x - 1}{e^y}$$

$$e^y\,dy = (e^x - 1)\,dx \quad\Rightarrow\quad \int e^y\,dy = \int (e^x - 1)\,dx$$

$$e^y = e^x - x + k \quad\Rightarrow\quad y = ln(e^x - x + k)$$

f) $x^2 y' + y^2 + 1 = 0$

$$y' = \frac{-1 - y^2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{-(1 + y^2)}{x^2} \Rightarrow \frac{dy}{1 + y^2} = \frac{-1}{x^2}\, dx$$

$$\int \frac{dy}{1 + y^2} = \int \frac{-1}{x^2}\, dx \Rightarrow arc\ tg\ y = \frac{1}{x} + k$$

$$y = tg \left(\frac{1}{x} + k\right)$$